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VARIABLE ORDER VARIABLE STEP FINITE DIFFERENCE METHODS FOR NONLINEAR BOUNDARY VALUE PROBLEMS

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1. INTRODUCTION

We intend to present in this paper a concise survey of some high order finite difference methods based on deferred corrections. The theoretical foundation, many of the tools, and some of the applications, have been developed in the last 8 years, and because of some recent results it seems a fitting time to recapitulate.

Basic notation and results are presented in Section 2 in a general though somewhat informal way.

The problem to which this work is addressed is stated in Section 3. In short it says: "Given a boundary value problem, a basic mesh region Ω_h, a discretization, and a tolerance, find an approximate solution defined (at least) in Ω_h and within the tolerance". A general scheme for solving this problem is also described in Section 3. Section 4 deals with applications and numerical results concerning nonlinear two-point boundary value problems, multipoint boundary value problems for first order nonlinear piecewise smooth systems, and finally some two-dimensional elliptic boundary value problems. In this last application the novelty lies in the coupling of fast direct methods with deferred corrections.

2. BASIC RESULTS AND NOTATION

In this Section we endeavor to present basic facts, that are already well known and will be needed subsequently, in a sufficiently general but simple notation which will serve more as a frame of reference than as an ultraformal hindrance. Thus we will not strive either to introduce the most general spaces and conditions possible or to be absolutely precise about all the details. There are two reasons for this choice of style: one is that the precise functional analysis formulations have been given earlier (Pereyra (1967b,c)), and the other is that in this paper we are mostly interested in calling the reader's attention to implementation and numerical performance.

We want to consider various boundary value problems for nonlinear differential equations, which we shall represent as

(2.1) $F(y) = 0$

with the domain and range of the operator F contained in appropriate function spaces. This is the continuous problem which, from now on, is assumed to have an unique solution y^*.

Finite differences will be used to provide discrete approximations Y defined only on a finite mesh region. A positive parameter h will be associated with the mesh, measuring in some way its maximum diameter. As usual, h is meant to tend to zero, and through the whole discussion we will consider $0<h<h_0$, for some fixed h_0.

The discrete problem will be denoted by

(2.2) $F_h(Y) = 0.$

In order to connect the continuous and discrete spaces of unknowns we introduce, for each h, a discretization mapping ϕ_h. With this mapping we can define the local truncation error $\tau_h(y^*)$:

(2.3) $\tau_h(y^*) \equiv F_h(\phi_h y^*).$

We assume that a unique solution Y(h) to (2.2) exists for each h.

We shall say that Y(h) converges discretely to y^* iff

(2.4) $\lim_{h\to 0} \| Y(h)-\phi_h y^* \| = 0.$

The discretization is consistent of order $p>0$ iff $\tau_h(y^*) = O(h^p)$. Stability is a notion that is inseparable from the two just given. It is well known that in the linear case stability of a discretization is simply implied by the uniform boundedness (with respect to h) of the inverse operator. In the nonlinear case a convenient way of stating the stability condition is the following: F_h is stable iff for any pair of discrete functions U,V, there exists a constant $c>0$, independent of h, such that

(2.5) $\| U-V \| \leq c \| F_h(U)-F_h(V) \|$.

A fundamental result (see for instance Pereyra (1967b) or (1973)) is the following:

Theorem 2.1 Let F_h be stable on the spheres $B_h \equiv B(\phi_h y^*, \rho)$ and consistent of order p with F. Then there exists an $h_0>0$ such that for each $0<h<h_0$ the discrete problem has an unique solution Y(h) which is convergent of order p to y^*, i.e. $\| Y(h)-\phi_h y^* \| = O(h^p)$.

The vector $e(h) = Y(h)-\phi_h y^*$ is called the global truncation error.

If a more detailed expression for the local truncation error is available in the form of an asymptotic expansion in powers of h then it is possible, under reasonable conditions, to obtain similar asymptotic expansions for the global error. In fact, let us assume that τ_h

can be expressed as

$$\tau_h = \sum_{j=p}^{J} F_j(y^*)h^j + O(h^{J+1}), \tag{2.6}$$

where the F_j are known operators (usually combinations of high order derivatives of y^*). Then, essentially under the assumptions of Theorem 2.1 and suitable smoothness requirements, there exists an expansion for $e(h)$:

$$e(h) = \phi_h \sum_{j=p}^{J} e_j h^j + O(h^{J+1}), \tag{2.7}$$

where the functions e_j are solutions of appropriate linear boundary value problems (see Stetter (1965), Pereyra (1967b,c), Lentini (1973) for proofs of this result in a general setting). If expansion (2.6) is lacunary in some regular way, for instance if only the terms F_{kp} are different from zero, then it is possible to show that expansion (2.7) will have a similar structure. Expansion (2.7) is basic for the method of Successive Extrapolations (Richardson extrapolation to h=0, Romberg integration), and in the survey papers of Joyce (1971) and Widlund (1971) a wealth of information and references about this important technique can be found. Given an initial mesh size h_0 the method of Successive Extrapolations (SE) requires the computation of the discrete solutions $Y(h_i)$, where h_i, $i=1,\ldots,I$, is a decreasing sequence such that the corresponding meshes M_i are refinements of the initial mesh M_0 associated with h_0. Appropriate linear combinations of the values of the mesh functions at the common grid points will produce the successive elimination of the leading error terms in (2.7). Accurate discrete solutions are then obtained on the coarsest mesh M_0 at the cost of solving equation (2.2) on much finer meshes. For one equation in one dimension this added cost can probably be offset by the simplicity of the algorithm. For systems of equations or in higher dimensions the storage and time requirements grow considerably and therefore more efficient techniques become necessary.

2.1. Deferred corrections.

As early as 1947 Leslie Fox advocated a technique for increasing the accuracy of discrete solutions that he called "difference corrections". Through the years he and his collaborators have applied this technique to a variety of problems in differential and integral equations as is attested in the book edited by Fox (1962). See also Pereyra (1967c) for an extensive set of references and historical account.

In Fox (1962) we find the term "deferred corrections" used interchangeably with that of difference corrections. We have preferred to adopt the former in our work which is, at least in appearance, differ-

ence free.

Deferred corrections are based on the observation that heightened precision can be obtained from a lower order solution by approximating terms in the local truncation error and re-solving. The important features are that the same grid size h is used throughout, and the same low order, simple operator must be inverted at each step. Also, as is the case of Successive Extrapolations, an algorithm for estimating asymptotically the global truncation error is implicit in the method.

Let T_ℓ be the segment of the local truncation error containing the first ℓ terms in the sum (2.6). Let S_ℓ be an approximation to T_ℓ (on the grid points) satisfying

$$T_\ell \equiv \phi_h \sum_{j=p}^{\ell.p} F_j(y^*)h^j = S_\ell(\phi_h y^*)+O(h^{(\ell+1).p}). \tag{2.8}$$

Then the deferred correction procedure is described by:

0) Solve $F_h(Y) = 0$. Call $Y^{(0)}$ its solution.

k) Solve $F_h(Y) = S_k(Y^{(k-1)})$. Call its solution $Y^{(k)}$. Repeat for k=1,...

In Pereyra (1967b,c) it is proven in a fairly general setting that the mesh functions $Y^{(k)}$ determined by this procedure satisfy

$$Y^{(k)}-\phi_h y^* = O(h^{(k+1).p}). \tag{2.9}$$

As a matter of fact, the equations need not be solved exactly (an impossible feat in practice), and that is also contemplated in the results mentioned above.

Naturally, the practical application of this procedure depends upon our ability to obtain the approximations S_k to the local truncation error. In all of our implementations we have made the natural choice, i.e. since the terms of the expansion for the local error are high order derivatives of the solution y*, we consider numerical differentiation formulas in terms of function values at the mesh points. As we shall see, very often the derivative order and the power in h coincide for each term in the sum to be approximated and therefore one of the dangers of numerical differentiation is not present. In true fact we are approximating differences of y* (remember Fox !), but our approach gives us a great deal of flexibility in automating all the procedures in a very efficient and accurate way. In Björck and Pereyra (1970), Galimberti and Pereyra (1970, 1971) and in Pereyra and Scherer (1973) a variety of algorithms and software are developed to cope, in particular, with numerical differentiation in one and several variables, both with Lagrangian and Hermite formulas. Incidentally, some of those algorithms could be used with profit in other tasks relevant

to this Conference, notably the computation of finite elements and of weights for numerical quadratures.

We would like to stress that formula (2.9) shows a remarkable property of deferred corrections, i.e. for a method of order p, p orders can be gained per correction. This fact was first observed by Daniel (see Daniel, Pereyra and Schumaker (1968)), and it can be seen at work in Pereyra (1968, 1973) , and in Section 4.1 of this paper, where a method of order four is used for two-point boundary value problems. A generalization to mildly nonlinear equations on rectangular regions using as a basic method a nine point formula and a newly developed fast direct solver is being completed (Concus, Golub and Pereyra (1973)).

For nonlinear problems, if it is desired to perform only one correction then a very economical procedure is available, specially if the nonlinear discrete equations are being solved by Newton's method. In fact, in that case, a program module should be available for computing the Jacobian matrix $F_h'(Y)$ and for solving linear systems of the form $F_h'\Delta = \beta$. It turns out that the following procedure will double the asymptotic order of our algorithm by a price equivalent to one Newton step and the computation of S_1:

0) Solve $F_h(Y) = 0$ for $Y^{(0)}$.

1) Solve the linear problem $F_h'(Y^{(0)})\Delta = S_1(Y^{(0)})$.

2) Correct: $Y^{(1)} = Y^{(0)} - \Delta$.

Then $Y^{(1)} - \phi_h y^* = O(h^{2p})$.

2.2. Asymptotic error estimates via deferred corrections.

Whenever a method to improve the accuracy exists, not far behind there must be one to estimate the error of the unimproved solution. In our case it is possible to prove (Pereyra (1970, 1973)) that the solution Δ_k to the linear problem

$$F_h'(Y^{(k)})\Delta = S_{k+1}(Y^{(k)}) - S_k(Y^{(k-1)}) \tag{2.10}$$

satisfies

$$\phi_h y^* - Y^{(k)} = \Delta_k + O(h^{p(k+1)}). \tag{2.11}$$

In other words, Δ_k is an asymptotic estimator for the error in $Y^{(k)}$ (or it can be used as a correction if that is the last step to be performed as we did in the last Section).

This error estimator can be used in a variety of ways besides the obvious one of reporting on line the accuracy of the computed solutions. In the applications to be described below we compare $\|\Delta_k\|$ with

$||\Delta_{k-1}||$ in order to take decisions about continuing corrections, decreasing the step, or interrupting the computation. It is clear that, since the vector Δ_k gives precise information about the error at each mesh point, that local step adjustments could be based on it.
The knowing reader will be aware by now that many of these ideas have been implicitly or explicitly advocated by Dahlquist and Henrici (1962), though with the exception of the initial value problem for ordinary differential equations we know of no systematic use of them in practical applications. The aim of this work is to call attention upon the possibility of constructing software for boundary value problems of similar characteristics to that available for initial value problems (cf. Gear (1971), Krogh (1969a, 1969b), and Bulirsch and Stöer (1966) as implemented by N. Clark (see Crane and Fox (1969))), and we shall proceed to expound in the next Sections some of the results of our efforts.

3. A GENERAL SCHEME

Based on the notation developed in Section 2 we present now a skeleton of the algorithm used in most of the applications to be described in Section 4. The problem to which the algorithm is addressed is the following:
"Given problem (2.1), a mesh region Ω_h (described shortly by the parameter h), a tolerance TOL>0, and discretization (2.2), find an approximate solution Y defined (at least) on Ω_h satisfying $||Y-\phi_h y^*|| <$TOL".
The basic mesh region Ω_h is that in which the user wants the approximate solution to be defined (minimal description).

Starting with Ω_h and the basic discrete method (2.2) an approximate solution is computed and an asymptotic error estimate is produced. If necessary, the net would have been refined in order to make that first step possible. Whenever the tolerance TOL is met, according to the error estimate, the process is interrupted. If the required accuracy has not been achieved then (2.2) is re-solved with the already computed correction vector $S_1(Y^{(0)})$ added in the right hand side. $S_2(Y^{(1)})$ is computed and used to produce a new error estimate, which is then compared with the former step error estimate. In order to continue the corrections on this mesh it is required that a significant gain in accuracy had been obtained, otherwise the mesh is refined. This last step acts exactly as the traditional sentinel on "the growth of high order differences", except that it takes into account other more complicated factors and that, of course, no differences must ever be inspected.

Whenever the mesh is refined the process restarts, with the difference that a fairly precise initial approximation is obtained by interpolation of the available approximate solution on the coarser mesh.
That this procedure is feasible and effective in a variety of non-trivial applications is borne out by the numerical results presented in the next Section and in the references.

4. APPLICATIONS

4.1. Special two-point boundary value problem.

The simple problem

$$\begin{aligned} &y''-f(x,y) = 0 \\ &y(a) = \alpha,\ y(b) = \beta \\ &f_y > -(\Pi/(b-a))^2 \end{aligned} \tag{4.1}$$

is one of the battle horses of many papers when testing hour arrives. Though the results of Section 4.2 will supersede completely those presented here, for historical and comparative reasons we would like to mention some of the results obtained with an implementation of the fourth order method

$$\begin{aligned} &h^{-2}(-Y_{i-1}+2Y_i-Y_{i+1})+\tfrac{1}{12}(f_{i-1}+10f_i+f_{i+1}) = 0\ , \quad i=1,\ldots,n-1 \\ &Y_0 = \alpha,\quad Y_n = \beta, \end{aligned} \tag{4.2}$$

on the uniform mesh $x_i = a+ih$, $i=0,\ldots,n$, $h = (b-a)/n$.
For smooth f the asymptotic expansion for the local truncation error is

$$\tau_h(y^*) = \phi_h \sum_{j=2}^{J} a_j f^{(2j)}(x,y^*(x))\frac{h^{2j}}{(2j)!} + O(h^{2J+2}) \tag{4.3}$$

with $$a_j = \frac{1}{(j+1)(2j+1)} - \frac{1}{6}\ .$$

At essentially the same cost as the simplest $O(h^2)$ method an $O(h^8)$ method can be obtained by approximating the first two terms in the sum (4.3) and performing one linearized deferred correction as indicated at the end of Section 2. Details on how the correction terms are actually computed are beyond the scope of this paper but have been given earlier in Pereyra (1965, 1967a, 1968, 1973).
An adaptive method (M2) based on the discretization (4.2) and following the scheme described in Section 3 has also been implemented.
Method M3 is an implementation of successive extrapolations. More details and Fortran programs for these methods are available in Pereyra (1973).
Our numerical experience indicates that these methods should be preferred (when they apply) to the general method of Section 4.2, since

they are usually more efficient and economical.

Example 1: $-y''+e^{y} = 0$,

$y(0) = y(1) = 0$.

Exact solution: $y^*(x) = -\ln 2+2 \ln (c \sec(\frac{c}{2}(x - \frac{1}{2})))$,

where the constant c satisfies: $c \sec \frac{c}{4} = \sqrt{2}$. The constant c is to 16 significant digits c = 1.336055694906108...

Whenever possible, we shall report the most accurate results obtained, which will be limited by the word length of the computer used, i.e. ∿16 decimal digits on IBM System 360 "double precision". Paraphrasing Keller (1972), this by no means signifies that our algorithms are only useful for these highly accurate solutions, since more modest accuracy can be obtained very economically on fairly coarse meshes.

The present problem has been chosen because of its frequent appearance in the literature of high order methods (cf. Perrin, Price, and Varga (1969), Ciarlet, Schultz, and Varga (1967), Herbold, Schultz, and Varga (1969), Jerome, and Varga (1969), Keller (1972), and Pereyra (1973)).

TABLE 1

Method	Initial mesh	Final mesh	Exact max. abs. error	Theoretical final order of the method
M1	32	32	4.08,-15	8
M2	8	16	5.2 ,-15	12
M3	8	64	2.5 ,-16	10

We point out that the estimated max. abs. error for M2 was equal to 5.5,-15, predicting very accurately the true error.

4.2. Multipoint boundary value problems for nonlinear first order systems.

Following Keller (1969, 1972) we consider multipoint boundary value problems of the form

$$y'(t)-f(t,y(t)) = 0 , \quad a<t<b, \tag{4.4}$$
$$g(y(\tau_1),\ldots,y(\tau_N)) = 0 ,$$

where y, f, and g are n-dimensional vector functions. Under suitable conditions Keller has proven that if $y^*(t)$ is an isolated solution of (4.4) then the discrete scheme known as the Box or centered-Euler scheme will provide an $O(h^2)$ approximation which can be computed by Newton's method. The error will have an asymptotic expansion in even powers of h, provided the data is piecewise smooth and the jump discontinuities occur at the boundary points. These results apply as well to the trapezoidal rule

$$(4.5) \qquad \frac{1}{h_j}(Y_j - Y_{j-1}) - \frac{1}{2}(f(t_{j-1},Y_{j-1}) + f(t_j,Y_j)) = 0$$

which is the one we have chosen in our experimentation. The work I am reporting in this Section is joint work with M. Lentini (cf. Lentini and Pereyra (1973), and Lentini (1973) where more details can be found). The nets considered by Keller are completely non-uniform, and eventually we could implement a variable order, variable step, deferred correction procedure of that generality. Since our work here constitutes one of the first steps in this direction (cf. Keller (1969), p. 22) and since further results require a major programming effort, we have limited somewhat the generality to the following simpler problem (cf. Keller (1968)):

$$(4.6) \qquad \begin{aligned} y'(t) - f(t,y(t)) &= 0 \ , \\ Ay(a) + By(b) &= \alpha \ , \end{aligned}$$

for which we use only uniform meshes.
However, we shall also present some preliminary results for a case in which there is an internal point of discontinuity, showing clearly that deferred corrections can be applied in such situations. This is a major departure from earlier results and we expect to pursue it further. It is clear now that going from an implementation for (4.6) to one for (4.4) is not too difficult. We doubt that completely general meshes are called for, and probably we will only consider piecewise uniform ones, i.e. independent uniform subdivisions between consecutive boundary (or discontinuity) points. In such a case our program is practically the same we have at present, with a minor modification in the linear equation solver and an extra loop in the correction generator. Provided the corrections are made using information only on these subintervals, without straddling boundary points, there are no new difficulties in the theoretical part either.

The asymptotic expansion for the local truncation error corresponding to the discretization (4.5) is

$$(4.6') \qquad \tau_h = -\sum_{j=1}^{J} \frac{j}{2^{2j-1}(2j+1)} f^{(2j)}(x,y^*(x)) \frac{h^{2j}}{(2j)!} + O(h^{2J+2}) \ .$$

With a minor modification due to the fact that the values of Y at the end points are also unknown, the deferred correction generator of Pereyra (1973) can, and has been used, in our implementation.
It would be very advisable to develop a different program for second order systems when no first derivatives are involved, generalizing the method described in Section 4.1.

We remark that high order equations can be reduced to first order systems in the usual way, but that in sharp contrast with other high order methods, both the function and all its derivatives up to one unit less than the order of the equation are obtained with the same accuracy.

Example 2 (Ciarlet, Schultz, and Varga (1967)).

$$y^{(IV)}(x) = (x^4+14x^3+49x^2+32x-12)e^x ,$$

$$y(0) = y'(0) = y(1) = y'(1) = 0 .$$

Exact solution: $y^*(x) = x^2(1-x)^2e^x$.

Starting with a uniform grid containing 9 points (including the boundary points) automatic subdivision and corrections furnished the following results:

TABLE 2

	Exact max. error in $y(x)$	Exact max. error in $y'(x)$
Ciarlet, Schultz, Varga (1967) (best results)	7.614,-5	2.130,-3
This method 9 points 2 correct.	1.975,-5	2.893,-5
This method 33 points 6 correct.	2.162,-15	1.443,-15

Again the errors were estimated very precisely.

Example 3. A problem with one jump discontinuity (Perrin, Price, and Varga (1968)):

$$y^{(IV)}(x) = \begin{cases} 24.0 , & 0 \le x \le 1/2 , \\ 48.0 , & 1/2 < x \le 1 , \end{cases}$$

with the same boundary conditions of Example 2.

Exact solution:

$$y^*(x) = \begin{cases} x^4 - \frac{19}{8}x^3 + \frac{21}{16}x^2 , & 0 \le x \le 1/2 , \\ 2(x-1)^4 + \frac{29}{8}(x-1)^3 + \frac{27}{16}(x-1)^2 , & 1/2 < x \le 1 . \end{cases}$$

In Perrin, Price, and Varga (1968) a variational method using cubic Hermite test functions is compared with a finite difference method that consists of a five points, $O(h^2)$ discretization of $y^{(IV)}(x)$. The variational method shows beautifully the expected fourth order approxi-

mation, while the finite difference method shows a very erratic behavior. The best results for the variational method are obtained with a Hermite subspace of dimension 18 and the max. abs. error on [0,1] is 1.25×10^{-5}, while for the finite difference method the best results on a mesh with 80 points give a max. abs. error on the grid points of 8.53×10^{-3}. These results seem to discredit all finite difference methods.

However, it is clear from our brief discussion earlier that (a) the finite difference approximation straddles the discontinuity for grid points in a neighborhood of x = 1/2, while the two point Hermite approximation does not, (b) in the best of cases, the finite difference approximation will only have order $O(h^2)$.

What happens if one uses an appropriate (though not special!) finite difference method?

In Table 3 we present the results obtained with our adaptive method. Again we list max. abs. errors on the grid points and also for each correction, the computed order (O_k) of the method obtained comparing the errors for two successive meshes. N is the total number of mesh points, including the end points, and k is the correction number.

TABLE 3

N\k	0	O.	1	O.	2	O.	3	O.
9	6.05,-3	-	4.43,-6	-	-	-	-	-
17	1.53,-3	2.0	2.75,-7	4.0	1.08,-9	-	4.22,-12	-
33	3.82,-4	2.0	1.72,-8	4.0	1.68,-11	6.0	1.65,-14	8.0
65	9.56,-5	2.0	1.07,-9	4.0	2.62,-13	6.0	6.94,-17	7.9

Thus we see that finite differences, properly used, can handle this type of problem and can produce truly highly accurate solutions. Also it is important to remark that again the error in the approximations to $y^{(i)}$, i = 0,1,2,3 was predicted very accurately and that all these derivatives were approximated with the same relative error. Observe that the method without corrections (column 0) shows a perfect non-erratic behavior, as predicted by the theory. (So do the corrected ones by the way.) These results were obtained in 78.17 sec. of computing time on an IBM 360/50 computer, using 138 K bytes of main memory. By eliminating the printing this time can be considerably reduced.

4.3. Poisson, Helmholtz and mildly non-linear elliptic equations. Rectangular regions.

The linear equations considered are

(4.7a) $-\Delta u+f(x,y) = 0$,

(4.7b) $-\Delta u+\lambda u+f(x,y) = 0$,

with Dirichlet boundary conditions on a rectangle of the plane (x,y).

The Laplacian operator is discretized on an uniform mesh by the usual 5-point formula (Δ_h).

Combining the fast linear equation solver of Buneman (1969) with deferred corrections provides a very effective, highly accurate procedure for these multidimensional problems.

The asymptotic expansion corresponding to Δ_h is simply (cf. Pereyra (1967a))

$$\tau_h = \sum_{j=1}^{J} \frac{2}{(2j+2)!} \left(\frac{\partial^{2j+2}u}{\partial x^{2j+2}} + \frac{\partial^{2j+2}u}{\partial y^{2j+2}}\right) h^{2j} + O(h^{2J+2}) \quad . \tag{4.8}$$

Again the deferred correction generator of Pereyra (1973) can be used for obtaining the corrections in each coordinate direction (cf. Pereyra (1970) for more details and generalizations).

Example 4.

$-\Delta u-50 \sin(5(x+y)) = 0$, on the unit square.

Exact solution (and boundary function)

$u^* = \sin(5(x+y))$

We give in Table 4 the exact and estimated max. relative errors on a 32x32 grid for several corrections.

TABLE 4

Correction	estimated max. rel. error	exact max. rel. error
0	0.698,-1	0.652,-1
1	0.936,-4	0.809,-4
2	0.119,-4	0.127,-4
3	0.922,-6	0.844,-6
4	0.379,-7	0.775,-7

These results were obtained on the IBM 360/67 computer at the Stanford University Computing Center. The execution time was 49.94 seconds, using a WATFIV program. Observe that the factor 5 in the argument of the sin function gives a solution with large derivatives. For instance if we consider the problem $-\Delta u-2 \sin(x+y) = 0$, whose solution is $u^*(x,y) = \sin(x+y)$, then in four corrections a precision of $0.46\text{x}10^{-11}$ is achieved with the same computing time. The five point formula alone (i.e., with no corrections) on a 64x64 grid gives in this case a precision of $0.29\text{x}10^{-5}$ in 17.12 seconds of computing time. Due to memory restrictions the largest mesh that could be processed with this program is 512x512. Extrapolating from the results for coars-

er meshes the $O(h^2)$ method would give an accuracy of $.46 \times 10^{-9}$ in about 1649 seconds of computing time. Results for the Helmholtz equation (4.7b) are similar. Details of this work will be published elsewhere. Coupling the techniques of Concus and Golub (1972) with deferred corrections we can extend these results to mildly nonlinear elliptic equations. The nonlinear equations are solved via a relaxed Picard iteration and a fast Helmholtz solver. Preliminary experimentation seems to indicate that in this case it is better to use a fix length correction formula from the beginning in order to diminish the number of nested iterations. It would be interesting to use a Newton type iteration, but in that case the use of fast solvers seems to be precluded. Instead of using the five point formula as a basic method one can use an $O(h^4)$ nine point formula (cf. Kantorovich and Krylov (1958)), for which a fast solver has just been developed by G.H. Golub. In this joint work with P. Concus and G.H. Golub we plan to extend to two dimensions the method of Section 4.1.

4.4. Poisson's equation on curved regions (Pereyra and Widlund (1973)).

For general regions the theoretical and practical difficulties of high order finite difference methods are considerably increased. To start with, the existence of asymptotic expansions depends very much upon the treatment of the boundary (cf. Pereyra (1967a)). A few years ago H.O. Kreiss proposed a method which, by using high order Lagrange interpolation at the boundary would guarantee the existence of expansions in even powers of h up to order h^6. Details of this result were never published and in the paper mentioned in the heading we shall describe the method in detail, give a complete proof of the necessary results, and perform deferred corrections. As a preview we offer here some numerical results.

Example 5:

$-\Delta u - c^2(\sin cx + \sin cy) = 0$, on the circle of radius 1/2 and center at (0,0).

Exact solution (and boundary function).

$u^*(x,y) = \sin cx + \sin cy$.

For h = 1/28 we obtained the following max. abs. errors for various values of c

Correction	c=1	c=0.1	c=10
0	8.9,-7	9.0,-12	2.5,-2
1	1.2,-10	3.6,-14	2.8,-4
2	2.4,-11	3.4,-14	6.3,-5
3	6.1,-12	4.0,-14	2.9,-5

The high order boundary interpolation is essential. Experiments show that when it is not performed no improvements are obtained with the corrections.

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REFERENCES

1. Björck, Å. and V. Pereyra, "Solution of Vandermonde systems of equations". Math. Comp. 24, 893-903 (1970).
2. Bulirsch R. and J. Stöer, "Numerical treatment of ordinary differential equations by extrapolation methods". Numer. Math. 8, 1-13 (1966).
3. Ciarlet, P.G., M.H. Schultz, and R.S. Varga, "Numerical methods of high-order accuracy for nonlinear boundary value problems I". Numer. Math. 9, 394-430 (1967).
4. Concus P., and G.H. Golub, "Use of fast direct methods for the efficient numerical solution of nonseparable elliptic equations". Techn. Rep. STAN-CS-72-278, Computer Sciences Dept., Stanford Univ. California (1972).
5. Concus P., G.H. Golub and V. Pereyra, "Fast direct deferred correction solver for mildly nonlinear elliptic equations on rectangular regions". To appear.
6. Crane, P.C. and P.A. Fox, "DESUB-Integration of a first order system of ordinary differential equations". Numerical Math. Comp. Programs 2, Issue 1. Bell Labs., New Jersey (1969).
7. Daniel J.W., V. Pereyra and L.L. Schumaker, "Iterated deferred corrections for initial value problems". Acta Cient. Venezolana 19, 128-135 (1968).
8. Falkenberg, J.C., "A method for integration of unstable systems of ordinary differential equations subject to two-point boundary conditions". BIT 8, 86-103 (1968).
9. Fox, L. (Editor) Numerical Solution of Ordinary and Partial Differential Equations. Pergamon Press, Oxford (1962).

10. Galimberti G. and V. Pereyra, "Numerical differentiation and the solution of multidimensional Vandermonde systems". Math. Comp. 24, 357-364 (1970).

11. _____, "Solving confluent Vandermonde systems of Hermite type". Numer. Math. 18, 44-60 (1971).

12. Gear, C.W., Numerical Initial Value Problems in Ordinary Differential Equations. Prentice Hall, New Jersey (1971).

13. Henrici, P., Discrete Variable Methods in Ordinary Differential Equations. Wiley, New York (1962).

14. Herbold, R.J., M.H. Schultz, and R.S. Varga, "The effect of quadrature errors in the numerical solution of boundary value problems by variational techniques". aequat. math. 3, 247-270 (1969).

15. Holt, J.F., "Numerical solution of nonlinear two-point boundary problems by finite difference methods". Comm. A.C.M. 7, 366-373 (1964).

16. Jerome, J.W. and R.S. Varga, "Generalizations of spline functions and applications to nonlinear boundary value and eigenvalue problems". In Theory and Applications of Spline Functions, 103-155. Academic Press, New York (1969).

17. Joyçe, C.C., "Survey of extrapolation processes in numerical analysis". SIAM Rev. 13, 435-490 (1971).

18. Kantorovich, L.V. and V.I. Krylov, Approximate Methods of Higher Analysis. Noordhoff, The Netherlands (1958).

19. Keller, H.B., "Accurate difference methods for nonlinear two-point boundary value problems". Manuscript (1972).

20. Keller, H.B., "Accurate difference methods for linear ordinary differential systems subject to linear constraints". SIAM J. Numer. Anal. 6, 8-30 (1969).

21. _____, Numerical Methods for Two-Point Boundary-Value Problems. Blaisdell, Mass. (1968).

22. Kreiss, H.O. "Difference approximations for ordinary differential equations". Uppsala Univ. Comp. Sc. Dept. Rep. NR 35 (1971).

23. Krogh, F.T., "A variable step variable order multistep method for the numerical solution of ordinary differential equations". Proceeding IFIP-68, 194-199. North-Holland Pub. Co., Amsterdam (1969a).

24. _____, "VODQ/SVDQ/DVDQ-Variable order integrators for the numerical solution of ordinary differential equations". TU Doc. CP-2308, NPO-11643, Jet Propulsion Lab., Pasadena, Cal. (1969b).

25. Lentini, M., "Correcciones diferidas para problemas de contorno en sistemas de ecuaciones diferenciales ordinarias de primer orden".

Pub. 73-04, Depto. de Comp., Fac. Ciencias, Univ. Central de Venezuela, Caracas (1973).

26. Lentini M. and Pereyra V., "A variable order variable step finite difference method for non-linear multipoint boundary value problems". To appear.
27. Osborne, M.R., "On shooting methods for boundary value problems". J. Math. Anal. App. 27, 417-433 (1969).
28. Pereyra V., "The difference correction method for nonlinear two-point boundary value problems". Stanford Univ. CS18, California (1965).
29. _____ , "Accelerating the convergence of discretization algorithms". SIAM J. Numer. Anal. 4, 508-533 (1967a).
30. _____ , "Iterated deferred corrections for nonlinear operator equations". Numer. Math. 10, 316-323 (1967b).
31. _____ , "Highly accurate discrete methods for nonlinear problems". MRC Techn. Rep. 728, Univ. of Wisconsin, Madison (1967c).
32. _____ , "Iterated deferred corrections for nonlinear boundary value problems". Numer. Math. 11, 111-125 (1968).
33. _____ , "Highly accurate numerical solution of casilinear elliptic boundary value problems in n dimensions". Math. Comp. 24, 771-783 (1970).
34. _____ , "High order finite difference solution of differential equations". Stanford Univ. Report, California (1973).
35. Pereyra V. and G. Scherer, "Efficient computer manipulation of tensor products with application to multidimensional approximation". To appear in Math. Comp. (1973).
36. Pereyra V. and O. Widlund, "A sixth order method for mildly nonlinear elliptic equations on curved boundaries". To appear.
37. Perrin, F.M., H.S. Price, and R.S. Varga, "On high-order numerical methods for nonlinear two-point boundary value problems". Numer. Math. 13, 180-198 (1969).
38. Roberts, S.M., J.S. Shipman, and W.J. Ellis, "A perturbation technique for nonlinear two-point boundary value problems". SIAM J. Numer. Anal. 6, 347-358 (1969).
39. Stetter, H., "Asymptotic expansions for the error of discretization algorithms for nonlinear functional equations". Numer. Math. 7, 18-31 (1965).
40. Widlund, O., "Some recent applications of asymptotic error expansions to finite-difference schemes". Proc. Roy. Soc. London A323, 167-177 (1971).