SMOOTH MONOTONE SPLINE INTERPOLATION

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1. INTRODUCTION

In the course of designing an automatic mesh refinement procedure for solving two-point boundary value problems for ordinary differential equations, the authors [3] came accross the need to produce smooth monotonic interpolating functions associated with monotonic data.

The literature offered some results and procedures [2,4] , and in more recent times (i.e. after this research had been completed) we also found [1,5] . However,none of these results fulfilled all of our requirements.

In this paper we present some new algorithms for performing this task, which are modifications and improvements of those stated in [2] . These algorithms show how to construct a piecewise linear, monotone interpolant, and then they use piecewise Bernstein polynomials in order to produce high order splines.

The algorithms are simple, and of the marching or local type. Thus, no system of equations have to be solved, and local knot adjustements can be made adaptively, for instance in order to inforce some bound restrictions on high order derivatives.

Even more recently Roulier [6] has also produced algorithms for shape preserving interpolation which are similar to the ones we offer.

2. PRELIMINARY RESULTS

In this section we shall give some definitions and results due to Mc Allister, Passow, and Roulier [2] , that will be used in the sequel. Let $\pi = \{x_0 < x_1 < \dots < x_N\}$ and let us consider the set of data pairs $\Delta = \{(x_i , y_i) , i=0,1,\dots,N\}$.

<u>Definition 2.1</u> The data in Δ are non-decreasing iff $y_i \leq y_{i+1}$, $i=0,\dots,N-1$. If all the inequalities are strict, then the data are increasing.

We consider now the successive slopes $M_i = (y_{i+1}-y_i)/(x_{i+1}-x_i)$ $i=0,\ldots,N-1$.

Definition 2.2 The data in Δ are non-concave iff $M_{i-1} \leq M_i$, $i=1,\ldots,N-1$. If all the inequalities are strict, then the data are convex. If the inequalities are reversed, then we say that the data are non-convex and concave, respectively.

We shall use the generic name of "monotone" for data that has either one of the properties : increasing, decreasing, non-increasing, or non-decreasing.

Let $0 < \alpha < 1$, and consider the points $\bar{x}_i = x_{i-1} + \alpha h_i$, with $h_i = x_i - x_{i-1}$. Let $\bar{\pi} = \{\bar{x}_i\}$. We are interested in generating piecewise linear interpolants of the data Δ, with breakpoints at the $\bar{x}_i$.

Definition 2.3 We shall say that the set of real numbers $\{\bar{y}_i\}$, $i=1,\ldots,N$, is monotone α-admissible, iff the piecewise linear function $L(x)$ constructed by joining the points $(x_0,y_0),(\bar{x}_1,\bar{y}_1),\ldots,(\bar{x}_N,\bar{y}_N),(x_N,y_N)$ interpolates the data Δ and is itself monotone.

We will denote by $\sum_n^m(\pi)$ the set of splines of degree n and deficiency n-m on π. That is $\varphi \in \sum_n^m(\pi)$ is a polynomial of degree at most n in each subinterval $[x_{i-1}, x_i]$ and $\varphi \in C^m[x_0,x_N]$. Passow and Roulier [4] have given conditions for the existence of α-admissible sets. We state here a somewhat less general result than that of [4] , which will be sufficient for our purposes.

Theorem 2.4 Let p,n be positive integers satisfying $p < n$ and let $\alpha = p/n$. Let $m = \min(p, n-p)$. Then there exist sets $\{\bar{y}_i\}$ monotone (convex) α-admissible for monotone (convex) data $\Delta = \{x_i,y_i\}$ iff for each $1 \leq k$, there is a monotone (convex) spline $s \in \sum_{kn}^{km}(\pi)$ that interpolates the data Δ and that satisfies:
i) $s^{(j)}(x_i^+) = 0$, $j=2,\ldots,pk$; ii) $s^{(j)}(x_i^-) = 0$, $j=2,\ldots,(n-p)k$.

We observe then that for fixed even degree n, the maximum order of continuity is obtained when $p = n/2$, which implies that $\alpha = 1/2$. Since we are interested in smoothness for our interpolants, then we shall concentrate in this case from now on. All our results carry over to the more general case and even to the one considered in [2] .

Passow and Roulier also mention that, to obtain $s(x)$ constructively it is enough to consider $s(x) = S_i(x)$ on $[x_{i-1},x_i]$, where $S_i(x)$ is the Bernstein polynomial of degree kn associated with the restriction of the piecewise linear function $L(x)$ to

$[x_{i-1}, x_i]$. This is

(2.5) $$S_i(x) = \left(\sum_{\nu=0}^{kn} L(x_{i-1} + \nu(x_i - x_{i-1})/kn) \binom{kn}{\nu} (x - x_{i-1})^{\nu} (x_i - x)^{kn-\nu} \right) / (x_i - x_{i-1})^{kn}$$

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3. MONOTONE INTERPOLATION

In addition to Theorem 2.4, Mc Allister et al [2] give an independent criterium (and constructive test) for the existence of α-admissible sets associated with convex, increasing data. Unfortunately this α need not be 1/2, even for very innocent looking data. A further result says that $\alpha = 1/2$ can be obtained in the case that the third order differences of the data are non-negative.

In what follows we will show that if one drops the requirement of convexity in the resulting interpolant then he can obtain 1/2-admissibility, both for non-concave monotone and simply monotone data.

<u>Theorem 3.1</u> Given a non-decreasing, non-concave set of data $\{x_i, y_i\}$, $i=0,\ldots,N$, there exists a non-decreasing piecewise linear function $L(x)$ that has break points at $\bar{x}_i = x_i + 0.5\,(x_{i+1} - x_i)$ and satisfies $L(x_i) = y_i$.

<u>Proof</u> : Let $L_0(x) = M_0\,(x - x_1) + y_1$ and consider $L(x) = L_0(x)$ for $x_0 \leq x \leq \bar{x}_1$. Let $\bar{y}_1 = L_0(\bar{x}_1)$. We define $L_i(x)$, the ith linear segment of $L(x)$, as the one joining the pair of points $(\bar{x}_i, \bar{y}_i)$, (x_i, y_i) , i.e.

$$L_i(x) = (y_i - \bar{y}_i)(x - x_i)/(x_i - \bar{x}_i) + y_i$$

and take $L(x) = L_i(x)$ for $\bar{x}_i \leq x \leq \bar{x}_{i+1}$. Finally, let $\bar{y}_{i+1} = L_i(\bar{x}_{i+1})$

Assume now that we have constructed successfully our polygonal through the (i-1)-segment. In order to continue the construction of the monotone interpolant we must have $y_i \leq \bar{y}_{i+1} \leq y_{i+1}$.

It is easy to see that the worst case occurs for $\bar{y}_i = y_{i-1}$. But then we would have

$$\bar{y}_{i+1} = L_i(\bar{x}_{i+1}) = (\bar{y}_i - y_i)(\bar{x}_{i+1} - x_i)/(\bar{x}_i - x_i) + y_i = M_{i-1}(x_{i+1} - x_i) + y_i \ .$$

But, because of the non-concavity of the data

$$\bar{y}_{i+1} = M_{i-1}(x_{i+1} - x_i) + y_i \leq M_i(x_{i+1} - x_i) + y_i = y_{i+1} \ .$$

Finally, since $\bar{y}_i \leq y_i$, the slope of the ith segment is positive, and therefore it is obvious that $y_i \leq \bar{y}_{i+1}$. □

The other combinations of monotonic behavior with convexity or concavity can all be reduced to the case just considered by simple changes of the dependent and/or independent variables. For instance, non-decreasing, non-convex data (x_i, y_i) is transformed into the appropriate shape by considering $(-x_{n-i}, -y_{n-i})$.

If the data are non-concave (non-convex), it could happen that the construction of Theorem 3.1 fails. We show now how to modify the algorithm in the case that the data are only monotonic. The idea is to add some artificial data points.

<u>Theorem 3.2</u> For any set of non-decreasing data points (x_i, y_i), $i=0,...,N$, it is possible to construct a non-decreasing, piecewise linear interpolant with break points at $\bar{x}_i = x_i + 0.5\,(x_{i+1}-x_i)$.

<u>Proof</u>: The algorithm proceeds as in Theorem 3.1, provided that $\bar{y}_{i+1} \leq y_{i+1}$. If this condition is violated, then we introduce one additional, auxiliary data point. Let us assume then, that $y_{i+1} < \bar{y}_{i+1}$. We introduce a new data pair (x^*, y^*) with $x_i < x^* < \bar{x}_{i+1}$, $y_i < y^* < y_{i+1}$, which, if chosen appropriately, will allow us to continue the construction of the monotone linear interpolant.

The auxiliary data point x^* is defined as the abscissa of the intersection of the segment $L_i(x)$ with the line $y = y_{i+1}$, i.e. $x^* = (y_{i+1} - y_i)/m_i + x_i$, where the slope $m_i = (y_i - \bar{y}_i)/(x_i - \bar{x}_i)$. The new mid-points are

$$\bar{x}^* = 0.5\,(x^* + x_i), \qquad \bar{x}^{**} = 0.5\,(x_{i+1} + x^*)$$

and the corresponding $\bar{y}^* = L_i(\bar{x}^*) = m_i(\bar{x}^* - x_i) + y_i < y_{i+1}$. We show now how to choose y^* so that $\bar{y}^* \leq y^* \leq y_{i+1}$, and also that $\bar{y}^{**} \leq y_{i+1}$. In fact, any y^* satisfying

$$\bar{y}^* \leq y^* \leq (x^* - \bar{x}^*)\,(y_{i+1} - \bar{y}^*)/(\bar{x}^{**} - \bar{x}^*)$$

will do, as is easily verified, and then the construction can be continued. □

We have implemented this procedures and they work well in our applications, as we shall report elsewhere [3].
It is also possible to produce shape preserving interpolants by using similar techniques.

REFERENCES

1. Fritsch, F.N. and R.E. Carlson "Monotone piecewise cubic interpolation". SIAM J. Numer. Anal. <u>17</u> : 238-246 (1980).
2. Mc Allister, D.F., E. Passow, and J.A. Roulier "Algorithms for computing slope preserving spline interpolation to data". Math.Comp. <u>31</u> : 717-725 (1977).
3. Pagallo, G. and V. Pereyra "Mesh selection by adaptive changes of variables". In preparation.
4. Passow, E.and J.A. Roulier "Monotone and convex spline interpolation". SIAM J. Numer. Anal. <u>14</u> : 904-909 (1977).
5. Pruess, S. "Alternatives to the exponential spline in tension".