On the Construction of Discrete Approximations to Linear Differential Expressions

By C. Ballester and V. Pereyra*

Introduction. When solving differential equations numerically by means of finite differences it is often necessary to obtain formulae for approximating linear combinations of derivatives. While classically these formulae have been given in terms of linear combinations of differences of the function at nodal points it is a current trend to consider directly formulae in terms of ordinates. Moreover, for use on a high-speed computer it is more convenient to be able to generate these formulae as needed instead of having to store them in the form of a table.

The main objective of this note is to describe an efficient algorithm for generating discrete approximations to linear differential expressions in terms of ordinates. This is done in Section 2 after the problem and its analytic solution have been stated in Section 1. In Section 3 we give two applications. One of them takes advantage of one of the features of the procedure which, in certain cases, makes the use of rational arithmetic quite simple.

1. Discrete Approximations to Linear Differential Expressions. Consider the mth-order homogeneous linear differential expression

$$L[y] = \sum_{\nu=0}^{m} f_\nu(x) y^{(\nu)} \tag{1}$$

in $y(x)$ with given continuous coefficients $f_\nu(x)$.

The linear combination

$$A(x) = \sum_{r=0}^{n} C_r y(x + \alpha_r h) \tag{2}$$

where C_r and α_r are constants, is called a *discrete approximation* of order p for (1) at the point $x = x_i$ if for any sufficiently differentiable function $y(x)$ in the interval containing the points x_i, $(x + \alpha_r h)$ $(r = 0, \cdots, n)$ the Taylor expansion of $A(x) - L[y](x)$ at x_i has its first nonzero term for $y^{(q)}(x_i)$, $q = m + p$.

It is shown in [3, pp. 161–162], that for any given $s \geqq 1$ and an arbitrary choice of $n + 1$ distinct points $\xi + \alpha_r h$ $(r = 0, \cdots, n)$, $n = m + s$, $n + 1$ quantities C_r can be found such that for every function $y(x)$ with $n + 1$ continuous derivatives

$$\sum_{r=0}^{n} C_r y(\xi + \alpha_r h) - L[y](\xi) = \frac{h^{s+m}}{(s+m)!} \sum_{p=0}^{n} \alpha_p^{\,n} C_p y^{(n)}(\eta) . \tag{3}$$

The coefficients C_k satisfy the Vandermonde system of equations

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$$\begin{aligned} \sum_{r=0}^{n} \alpha_r^{\,k} C_r &= \frac{k!}{h^k} f_k(\xi) \quad \text{for} \quad 0 \leqq k \leqq m\,, \\ &= \quad 0 \qquad\quad \text{for} \quad m+1 \leqq k \leqq n\,. \end{aligned} \tag{4}$$

There are applications in which several differential expressions of the type (1) have to be approximated by means of discrete formulae of diverse orders and with different configurations of nodal points (cf. Volkov [12], Pereyra [9]). In those cases it is of interest to solve the system of equations (4) in an efficient manner.

Closed formulae for the elements of the inverse of a Vandermonde matrix are well known (see for instance Gautschi [4], Macon and Spitzbart [8]). Traub [11] presents two algorithms for inverting Vandermonde matrices. The first of them seems to be the most efficient (in the number of operations required) of all algorithms known to us. By using the special structure of the matrix, Traub is able to reduce the number of operations to $\frac{1}{2}n(7n - 9)$ multiplications (divisions are counted as equal to multiplications) and $5n(n - 1)/2$ additions. Unfortunately, it is not clear to us how to obtain from this a similarly efficient method for solving systems of equations of the form (4) (with say $\sim 1/3$ less operations).

Lynnes and Moler [7] discuss an algorithm based on Neville's formula which solves a Vandermonde system with a number of operations proportional to n^3.

The aim of this note is to provide an algorithm for solving Vandermonde systems of linear equations which applies directly to the solution of (4) and for which the number of operations is proportional to n^2.

2. Solution of Vandermonde System of Linear Equations. Given the $n + 1$ real and distinct numbers $(\alpha_0, \cdots, \alpha_n)$, let

$$\mathbf{V}(\alpha_0, \cdots, \alpha_n) = (v_{ij}) \qquad (i, j = 0, \cdots, n) \tag{5}$$

be a Vandermonde matrix, i.e.

$$v_{ij} = \alpha_j^{\,i}\,. \tag{6}$$

We will derive an algorithm for solving the system of linear equations

$$\mathbf{V}(\alpha_0, \cdots, \alpha_n)\mathbf{x} = \mathbf{b}\,, \tag{7}$$

where $\mathbf{b} = (b_0, \cdots, b_n)$ is given. This algorithm will take into account the special structure of the matrix $\mathbf{V}$.

We claim that the factorization of the matrix $\mathbf{V}$ as a product of an upper and a lower triangular matrix can be given explicitly and in a very simple fashion. Observe that this factorization is possible, since all the principal minors of $\mathbf{V}$ are also of the Vandermonde type and, since the α_i are distinct, this implies that those minors are nonzero.

Let us consider the n bidiagonal matrices $\mathbf{L}^{(i)} = (l_{jk}^{(i)})$ whose nonzero elements are

$$\begin{aligned} l_{j,j}^{(i)} &= 1 \qquad\qquad (j = 0, \cdots, n) \\ l_{j,j-1}^{(i)} &= -\alpha_i \qquad (i = 0, \cdots, n-1)\,, \qquad (j = i+1, \cdots, n)\,. \end{aligned} \tag{8}$$

THEOREM. *Premultiplication of the system* (7) *by the matrices* $\mathbf{L}^{(0)}, \mathbf{L}^{(1)}, \cdots, \mathbf{L}^{(n-1)}$ *reduces it to upper triangular form. Moreover, this triangular form can be ex-*

plicitly written as $\mathbf{U} = (u_{ij})$ *with*

$$u_{0j} = 1 \qquad (j = 0, \cdots, n)$$
$$u_{ij} = 0 \qquad (i > j)$$
$$u_{ij} = \prod_{s=0}^{i-1} (\alpha_j - \alpha_s) \qquad (1 \leqq i \leqq j \leqq n) .$$

This last equation can also be written as

$$u_{ij} = (\alpha_j - \alpha_{i-1})u_{i-1,j} \qquad (1 \leqq i \leqq j \leqq n) .$$

Proof. The proof is by induction. Call $\mathbf{V}^{(0)} = \mathbf{V}(\alpha_0, \cdots, \alpha_n)$,

$$\mathbf{V}^{(i+1)} = \mathbf{L}^{(i)}\mathbf{V}^{(i)} . \tag{9}$$

Thus

$$\mathbf{V}^{(1)} = \mathbf{L}^{(0)}\mathbf{V}^{(0)} = (v_{ij}^{(1)}) \quad \text{where} \quad v_{ij}^{(1)} = \alpha_j^{i-1}(\alpha_j - \alpha_0)$$

for $i \geqq 1$, and $v_{0j}^{(1)} = 1$ $(j = 0, \cdots, n)$.

In particular, $v_{i0}^{(1)} = 0$ for $i \geqq 1$ and all the elements of the first column but the first one have been eliminated. We will show now that, more generally, $\mathbf{V}^{(k)}$ has the following form

$$\begin{aligned} v_{ij}^{(k)} &= v_{ij}^{(k-1)} && (0 \leqq i < k; 0 \leqq j \leqq n) , \\ v_{ij}^{(k)} &= \alpha_j^{i-k} \prod_{s=0}^{k-1} (\alpha_j - \alpha_s) && (k \leqq i \leqq n; 0 \leqq j \leqq n) . \end{aligned} \tag{10}$$

Formula (9) shows that (10) is true for $k = 1$. Assume that it is valid for k, $1 < k < n$. It is clear that multiplication by $\mathbf{L}^{(k)}$ does not disturb the first $k + 1$ rows and k columns of $\mathbf{V}^{(k)}$. On the other hand, for $k < i \leqq n$, $k \leqq j \leqq n$ we have

$$v_{ij}^{(k+1)} = v_{ij}^{(k)} - \alpha_k v_{i-1,j}^{(k)} , \tag{11}$$

and from (10)

$$\begin{aligned} v_{ij}^{(k+1)} &= \alpha_j^{i-k} \prod_{s=0}^{k-1} (\alpha_j - \alpha_s) - \alpha_k \alpha_j^{i-k-1} \prod_{s=0}^{k-1} (\alpha_j - \alpha_s) \\ &= \alpha_j^{i-k-1}(\alpha_j - \alpha_k) \prod_{s=0}^{k-1} (\alpha_j - \alpha_s) = \alpha_j^{i-k-1} \prod_{s=0}^{k} (\alpha_j - \alpha_s) \end{aligned}$$

which is (10) for the step $k + 1$. If we put $k = n$ in (10) we obtain

$$v_{0.j}^{(n)} = 1 , \qquad v_{i,j}^{(n)} = \prod_{s=0}^{i-1} (\alpha_j - \alpha_s) \qquad (1 \leqq i, j \leqq n) \tag{12}$$

and it is clear that all elements below the main diagonal are zero and thus

$$\mathbf{U} = (u_{ij}) = \mathbf{LV} = \left(\prod_{i=0}^{n-1} \mathbf{L}^{(i)}\right)\mathbf{V}(\alpha_0, \cdots, \alpha_n) \tag{13}$$

is an upper triangular matrix. Since $\mathbf{L}$ has ones on the main diagonal, this is the unique decomposition of $\mathbf{V}$ in terms of upper and lower triangular matrices having that property (cf. Householder [6]).

Observe that $\mathbf{L}$ has a very simple structure:

$$l_{i,i-k} = (-1)^k \sigma_{n,k}, \qquad (k = 1, \cdots, i; i = 2, \cdots, n)$$

where $\sigma_{n,k}$ is the kth elementary symmetric function of the (α_j). These $(n - 1)$ numbers can be constructed by the recursion

$$\sigma_{m,j} = \sigma_{m-1,j} + \alpha_m \sigma_{m-1,j-1} \qquad (m = 2, \cdots, n; j = 1, \cdots, m)$$
$$\sigma_{m,0} = 1, \qquad \sigma_{11} = \alpha_1 .$$

From (12) it follows that $\mathbf{U}$ can be constructed by means of the recursion formula

(14) $$u_{0j} = 1, \qquad u_{ij} = (\alpha_j - \alpha_{i-1}) u_{i-1,j} \qquad (1 \leqq i \leqq j \leqq n),$$

while the new right-hand side $\tilde{\mathbf{b}} = \mathbf{Lb}$ is to be obtained from:

(15) $$\mathbf{b}^{(0)} = \mathbf{b}, \qquad b_j^{(i)} = b_j^{(i-1)} - \alpha_{i-1} b_{j-1}^{(i-1)} \qquad (1 \leqq i \leqq j \leqq n)$$
$$\tilde{\mathbf{b}} = \mathbf{b}^{(n)} .$$

Once (14) and (15) have been computed the $\mathbf{x}$ in (7) can be obtained by the standard backward substitution

(16) $$x_s = \left(\tilde{b}_s - \sum_{i=s+1}^{n} u_{si} x_i \right) \Big/ u_{ss} .$$

If we consider division times as equivalent to multiplication times, then the algorithm (14)–(16) takes $\frac{1}{2}(3n + 2)(n + 1)$ multiplications and the same number of additions to produce the solution $\mathbf{x}$ of the system (7) with arbitrary right-hand side $\mathbf{b}$.

In Table 1 we give the number of operations corresponding to Traub's algorithm I (see [11]), and that of Lynnes and Moler [7]. In the former one we count also the operations involved in computing $\mathbf{x} = \mathbf{V}^{-1}\mathbf{b}$, after $\mathbf{V}^{-1}$ has been formed. Here $\mathbf{V}$ is a $n \times n$ matrix.

TABLE 1

	Traub (I)	*Lynnes & Moler*	B-P
$\times$	$9n(n - 1)/2$	$n(n + 1)(n + 2)$	$\frac{1}{2}n(3n - 1)$
$+$	$\frac{1}{2}n(7n - 5)$	$4n(n + 1)(n + 2)$	$\frac{1}{2}n(3n - 1)$

A warning should be given about the possible instability of all these algorithms. This is expected since Vandermonde matrices are known to be very ill conditioned (cf. [4]). For our particular algorithm we refer the reader to the comments of Wilkinson [13, Chapter 3, §26, p. 108] on the errors that occur when inverting triangular systems of equations by elimination.

3. Applications. In applying the method of iterated deferred corrections (cf. [9]) to the solution of the two-point boundary value problem

$$y'' = f(x, y) , \quad y(a) = \alpha , \quad y(b) = \beta , \tag{17}$$

it is necessary to construct discrete approximations to linear differential expressions like

$$L[y](x_i) = \sum_{j=1}^{N} h^{2j+2} \frac{2}{(2j+2)!} y^{(2j+2)}(x_i) , \qquad N = 1, 2, \cdots , \tag{18}$$

with orders $2N + 4$ in h at all the nodal points $x_i = a + ih$, $i = 1, \cdots, n - 1$, $n = (b - a)/h$.

It is clear from our definition that what we need are expressions of the form (2) of order $p = 2$, that is $q = 2N + 4$. Since we would like to use values of $y(x)$ only at the nodal points, it is clear that we can use symmetrical formulae if we stay far enough from the boundary. For points close to the boundary it will be necessary to use unsymmetrical formulae. Let us examine the simplest case, $N = 1$. A symmetrical formula with five points will give the required accuracy for $y^{(4)}$ and it can be obtained from Bickley's table [1], [2]:

$$L[y](x_j) = \frac{h^4}{12} y^{(4)}(x_j) = \frac{1}{12} \sum_{i=0}^{4} \binom{4}{i} (-1)^i y(x_j + (i-2)h) - \frac{h^6}{72} y^{(6)}(\xi) \qquad (j = 2, \cdots, n-2) \tag{19}$$

or else it can be generated by solving a system of five linear equations like (7), with

$$\alpha_i = i - 2 , \qquad b_i = 2\delta_{i2} \qquad (i = 0, \cdots, 4) . \tag{20}$$

For $j = 1$ and $j = n - 1$ we have to use unsymmetrical six-point formulae which still can be found in Bickley's table. If more terms in (18) are desired, then the boundary situation becomes more involved and the number of different formulae grows steadily. It is in this case that the use of our generating procedure becomes more advantageous, since by giving a few parameters we can obtain all the necessary coefficients quite rapidly. This is also true for more complicated situations like those arising in the solution of boundary value problems for partial differential equations.

As a second application of our procedure we have generated the coefficients for the n point approximations to the mth derivative of a function $y(x)$, for $m = 1(1)10$, $n = m + 1(1)11$ at every nodal point. This table superposes and complements that of Bickley, and a copy of it has been deposited in the UMT files. In the Microfiche Appendix we reproduce a part of it.

The procedure of Section 2 is particularly well suited for this purpose, since in this case the α_i are integers and the triangular decomposition does not change this situation. This can be clearly seen in formula (12). This means that only the backward substitution will have to be carried out in rational arithmetic in order to obtain the exact values of the coefficients. Another interesting feature, also shown in (12) is that the factorization tends to decrease the values of the entries. This has been proved to be of critical importance in the exact inversion of matrices with integer coefficients (cf. Rosser [10]), since otherwise the number of digits in intermediary calculations may grow beyond the word size of the computer being used.

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This table consists of the exact values of the integer coefficients ${}_{mn}A_{pr}$ and the coefficients to 5S (in floating-point form) of the error terms ${}_{mn}E_p$ for the discrete approximations

$$\frac{h^m}{m!}\ y^{(m)}(x_p) = \frac{1}{(n-1)!}\sum_{r=0}^{n-1} {}_{mn}A_{pr}\ y(x_r) + {}_{mn}E_p ,$$

where $x_r = x_0 + rh$, $p = 0(1)(n-1)$ for the ranges $m = 1(1)6$, $n=7,9$; $m= 5,6$, $n= 8,10$. The underlying calculations were performed on a CDC 3600 system.

An abridgement of Bickley's table [1] is given in the NBS Handbook [2]. The present authors have generated his entire table by the method of Gautschi [3] and thereby confirmed its accuracy.

The error term ${}_{mn}E_p$ can be expressed as ${}_{mn}e_p\ h^n\ y^{(n)}(\xi)$, where

$${}_{mn}e_p = -\frac{1}{n!\ (n-1)!}\sum_{j=0}^{n-1}(j-p)^n\ {}_{mn}A_{pj} .$$

For derivatives of even order the quantity ${}_{mn}e_{1/2(n-1)}$ vanishes, and the resulting symmetric formula is then accurate to an extra order of magnitude in h. Such error coefficients are identified in this supplementary table by an asterisk.

The authors include references to publications by Gregory [4] and Collatz [5]; however, they have not cited the relatively inaccessible tables of Kuntzmann [6,7], which contain similar information for the first 10 derivatives.

The coefficients ${}_{mn}A_{pr}$ $(r=0,\ldots,n)$ can be found in the column p of the table with heading: Derivative m, Points n. The Error row contains the coefficients ${}_{mn}e_p$ of the error terms.

J.W.W.

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DERIVATIVE 1
POINTS 8

P	0	1	2	3
A 0	-13068	-720	120	-48
A 1	35280	-7308	-1680	504
A 2	-52920	15120	-3948	-3024
A 3	58800	-12600	8400	-1260
A 4	-44100	8400	-4200	5040
A 5	21168	-3780	1680	-1512
A 6	-5880	1008	-420	336
A 7	720	-120	48	-36
ERROR	-1.2500-001	1.7857-002	-5.9524-003	3.5714-003

DERIVATIVE 1
POINTS 8

P	4	5	6	7
A 0	36	-48	120	-720
A 1	-336	420	-1008	5880
A 2	1512	-1680	3780	-21168
A 3	-5040	4200	-8400	44100
A 4	1260	-8400	12600	-58800
A 5	3024	3948	-15120	52920
A 6	-504	1680	7308	-35280
A 7	48	-120	720	13068
ERROR	-3.5714-003	5.9524-003	-1.7857-002	1.2500-001

DERIVATIVE 2
POINTS 8

P	0	1	2	3
A 0	13132	1764	-154	28
A 1	-56196	-980	2996	-378
A 2	110754	-6804	-5292	3780
A 3	-132860	11970	1820	-6860
A 4	103320	-9380	1190	3780
A 5	-50652	4536	-756	-378
A 6	14266	-1260	224	28
A 7	-1764	154	-28	0
ERROR	3.2411-001	-2.5893-002	4.6627-003	-8.9286-004

DERIVATIVE 2
POINTS 8

P	4	5	6	7
A 0	0	-28	154	-1764
A 1	28	224	-1260	14266
A 2	-378	-756	4536	-50652
A 3	3780	1190	-9380	103320
A 4	-6860	1820	11970	-132860
A 5	3780	-5292	-6804	110754
A 6	-378	2996	-980	-56196
A 7	28	-154	1764	13132
ERROR	-8.9286-004	4.6627-003	-2.5893-002	3.2411-001

DERIVATIVE 3
POINTS 8

P	0	1	2	3
A 0	-6769	-1624	-49	56
A 1	35728	6223	-1232	-497
A 2	-82509	-9744	4851	336
A 3	108920	8435	-7000	1715
A 4	-89075	-4760	5005	-3080
A 5	45024	1869	-2016	1869
A 6	-12943	-448	497	-448
A 7	1624	49	-56	49
ERROR	-3.2569-001	-3.4722-003	6.2500-003	-4.8611-003

DERIVATIVE 3
POINTS 8

P	4	5	6	7
A 0	-49	56	-49	-1624
A 1	448	-497	448	12943
A 2	-1869	2016	-1869	-45024
A 3	3080	-5005	4760	89075
A 4	-1715	7000	-8435	-108920
A 5	-336	-4851	9744	82509
A 6	497	1232	-6223	-35728
A 7	-56	49	1624	6769
ERROR	4.8611-003	-6.2500-003	3.4722-003	3.2569-001

DERIVATIVE 4
POINTS 8

P	0	1	2	3
A 0	1960	735	140	-35
A 1	-11655	-3920	-385	420
A 2	29820	8925	0	-1365
A 3	-42665	-11340	1085	1960
A 4	36960	8785	-1540	-1365
A 5	-19425	-4200	945	420
A 6	5740	1155	-280	-35
A 7	-735	-140	35	0
ERROR	1.6788-001	2.2049-002	-5.7292-003	1.2153-003

DERIVATIVE 4
POINTS 8

P	4	5	6	7
A 0	0	35	-140	-735
A 1	-35	-280	1155	5740
A 2	420	945	-4200	-19425
A 3	-1365	-1540	8785	36960
A 4	1960	1085	-11340	-42665
A 5	-1365	0	8925	29820
A 6	420	-385	-3920	-11655
A 7	-35	140	735	1960
ERROR	1.2153-003	-5.7292-003	2.2049-002	1.6788-001

DERIVATIVE 5
POINTS 8

P	0	1	2	3
A 0	-322	-175	-70	-7
A 1	2065	1078	385	-14
A 2	-5670	-2835	-882	189
A 3	8645	4130	1085	-490
A 4	-7910	-3605	-770	595
A 5	4347	1890	315	-378
A 6	-1330	-553	-70	119
A 7	175	70	7	-14
ERROR	-4.8611-002	-1.3889-002	-6.3657-004	1.3889-003

DERIVATIVE 5
POINTS 8

P	4	5	6	7
A 0	14	-7	-70	-175
A 1	-119	70	553	1330
A 2	378	-315	-1890	-4347
A 3	-595	770	3605	7910
A 4	490	-1085	-4130	-8645
A 5	-189	882	2835	5670
A 6	14	-385	-1078	-2065
A 7	7	70	175	322
ERROR	-1.3889-003	-6.3657-004	1.3889-002	4.8611-002

DERIVATIVE 6
POINTS 8

P	0	1	2	3
A 0	28	21	14	7
A 1	-189	-140	-91	-42
A 2	546	399	252	105
A 3	-875	-630	-385	-140
A 4	840	595	350	105
A 5	-483	-336	-189	-42
A 6	154	105	56	7
A 7	-21	-14	-7	0
ERROR	7.9861-003	3.8194-003	1.0417-003	-3.4722-004

DERIVATIVE 6
POINTS 8

P	4	5	6	7
A 0	0	-7	-14	-21
A 1	7	56	105	154
A 2	-42	-189	-336	-483
A 3	105	350	595	840
A 4	-140	-385	-630	-875
A 5	105	252	399	546
A 6	-42	-91	-140	-189
A 7	7	14	21	28
ERROR	-3.4722-004	1.0417-003	3.8194-003	7.9861-003

DERIVATIVE 5
POINTS 9

P	0	1	2	3	4
A 0	-4536	-1960	-560	0	56
A 1	32200	13104	3080	-560	-504
A 2	-100240	-38360	-7056	3080	1456
A 3	178920	64400	8680	-7056	-1624
A 4	-200480	-68040	-6160	8680	0
A 5	144536	46480	2520	-6160	1624
A 6	-65520	-20104	-560	2520	-1456
A 7	17080	5040	56	-560	504
A 8	-1960	-560	0	56	-56
ERROR	6.1863-002	1.3252-002	-6.3657-004	-6.3657-004	7.5231-004

DERIVATIVE 5
POINTS 9

P	5	6	7	8
A 0	-56	0	560	1960
A 1	560	-56	-5040	-17080
A 2	-2520	560	20104	65520
A 3	6160	-2520	-46480	-144536
A 4	-8680	6160	68040	200480
A 5	7056	-8680	-64400	-178920
A 6	-3080	7056	38360	100240
A 7	560	-3080	-13104	-32200
A 8	0	560	1960	4536
ERROR	-6.3657-004	-6.3657-004	1.3252-002	6.1863-002

DERIVATIVE 6
POINTS 9

P	0	1	2	3	4
A 0	546	322	154	42	-14
A 1	-4088	-2352	-1064	-224	168
A 2	13384	7504	3192	448	-728
A 3	-25032	-13664	-5432	-336	1624
A 4	29260	15540	5740	-140	-2100
A 5	-21896	-11312	-3864	448	1624
A 6	10248	5152	1624	-336	-728
A 7	-2744	-1344	-392	112	168
A 8	322	154	42	-14	-14
ERROR	-1.2500-002	-4.5139-003	-6.9444-004	3.4722-004	*7.5231-005

DERIVATIVE 6
POINTS 9

P	5	6	7	8
A 0	-14	42	154	322
A 1	112	-392	-1344	-2744
A 2	-336	1624	5152	10248
A 3	448	-3864	-11312	-21896
A 4	-140	5740	15540	29260
A 5	-336	-5432	-13664	-25032
A 6	448	3192	7504	13384
A 7	-224	-1064	-2352	-4088
A 8	42	154	322	546
ERROR	-3.4722-004	6.9444-004	4.5139-003	1.2500-002

DERIVATIVE 1
POINTS 10

P	0	1	2	3	4
A 0	-1026576	-40320	5040	-1440	720
A 1	3265920	-623376	-90720	19440	-8640
A 2	-6531840	1451520	-396576	-155520	51840
A 3	10160640	-1693440	846720	-223776	-241920
A 4	-11430720	1693440	-635040	544320	-72576
A 5	9144576	-1270080	423360	-272160	362880
A 6	-5080320	677376	-211680	120960	-120960
A 7	1866240	-241920	72576	-38880	34560
A 8	-408240	51840	-15120	7776	-6480
A 9	40320	-5040	1440	-720	576
ERROR	-1.0000-001	1.1111-002	-2.7778-003	1.1905-003	-7.9365-004

DERIVATIVE 1
POINTS 10

P	5	6	7	8	9
A 0	-576	720	-1440	5040	-40320
A 1	6480	-7776	15120	-51840	408240
A 2	-34560	38880	-72576	241920	-1866240
A 3	120960	-120960	211680	-677376	5080320
A 4	-362880	272160	-423360	1270080	-9144576
A 5	72576	-544320	635040	-1693440	11430720
A 6	241920	223776	-846720	1693440	-10160640
A 7	-51840	155520	396576	-1451520	6531840
A 8	8640	-19440	90720	623376	-3265920
A 9	-720	1440	-5040	40320	1026576
ERROR	7.9365-004	-1.1905-003	2.7778-003	-1.1111-002	1.0000-001

DERIVATIVE 2
POINTS 10

P	0	1	2	3	4
A 0	1172700	109584	-8028	1368	-324
A 1	-5973264	76860	189864	-21708	4608
A 2	15212448	-1041984	-284400	251424	-36288
A 3	-25357248	2062368	-78624	-448560	290304
A 4	29479464	-2344608	376488	208656	-516600
A 5	-24040800	1864296	-321552	31752	290304
A 6	13525344	-1028160	178416	-34272	-36288
A 7	-5012928	375264	-64800	14256	4608
A 8	1103868	-81648	14004	-3240	-324
A 9	-109584	8028	-1368	324	0
ERROR	2.8290-001	-1.9087-002	3.0357-003	-7.3413-004	1.5873-004

DERIVATIVE 2
POINTS 10

P	5	6	7	8	9
A 0	0	324	-1368	8028	-109584
A 1	-324	-3240	14004	-81648	1103868
A 2	4608	14256	-64800	375264	-5012928
A 3	-36288	-34272	178416	-1028160	13525344
A 4	290304	31752	-321552	1864296	-24040800
A 5	-516600	208656	376488	-2344608	29479464
A 6	290304	-448560	-78624	2062368	-25357248
A 7	-36288	251424	-284400	-1041984	15212448
A 8	4608	-21708	189864	76860	-5973264
A 9	-324	1368	-8028	109584	1172700
ERROR	1.5873-004	-7.3413-004	3.0357-003	-1.9087-002	2.8290-001

DERIVATIVE 3
POINTS 10

P	0	1	2	3	4
A 0	-723680	-118124	64	1324	-944
A 1	4581036	457560	-118764	-13176	10764
A 2	-13502376	-734544	460440	-59184	-55656
A 3	24383184	672504	-742224	301560	54096
A 4	-29570184	-422856	685944	-464184	103320
A 5	24743880	197064	-438984	352296	-226296
A 6	-14163576	-62160	210504	-160944	154056
A 7	5314896	11304	-69840	51624	-47664
A 8	-1181304	-684	14184	-10260	9144
A 9	118124	-64	-1324	944	-820
ERROR	-3.2316-001	2.3534-003	2.1770-003	-1.4716-003	1.1299-003

DERIVATIVE 3
POINTS 10

P	5	6	7	8	9
A 0	820	-944	1324	64	-118124
A 1	-9144	10260	-14184	684	1181304
A 2	47664	-51624	69840	-11304	-5314896
A 3	-154056	160944	-210504	62160	14163576
A 4	226296	-352296	438984	-197064	-24743880
A 5	-103320	464184	-685944	422856	29570184
A 6	-54096	-301560	742224	-672504	-24383184
A 7	55656	59184	-460440	734544	13502376
A 8	-10764	13176	118764	-457560	-4581036
A 9	944	-1324	-64	118124	723680
ERROR	-1.1299-003	1.4716-003	-2.1770-003	-2.3534-003	3.2316-001

DERIVATIVE 4
POINTS 10

P	0	1	2	3	4
A 0	269325	67284	6363	-1638	441
A 1	-1932084	-403515	3654	22743	-6048
A 2	6275052	1095696	-117180	-70056	42588
A 3	-12135312	-1799028	332136	79380	-122976
A 4	15403374	1994328	-462798	-11844	171990
A 5	-13287960	-1552194	390852	-50022	-122976
A 6	7770924	841680	-215964	46872	42588
A 7	-2962512	-303156	78120	-19404	-6048
A 8	666477	65268	-16821	4410	441
A 9	-67284	-6363	1638	-441	0
ERROR	1.9943-001	1.4010-002	-3.5246-003	9.8931-004	-2.2597-004

DERIVATIVE 4
POINTS 10

P	5	6	7	8	9
A 0	0	-441	1638	-6363	-67284
A 1	441	4410	-16821	65268	666477
A 2	-6048	-19404	78120	-303156	-2962512
A 3	42588	46872	-215964	841680	7770924
A 4	-122976	-50022	390852	-1552194	-13287960
A 5	171990	-11844	-462798	1994328	15403374
A 6	-122976	79380	332136	-1799028	-12135312
A 7	42588	-70056	-117180	1095696	6275052
A 8	-6048	22743	3654	-403515	-1932084
A 9	441	-1638	6363	67284	269325
ERROR	-2.2597-004	9.8931-004	-3.5246-003	1.4010-002	1.9943-001

DERIVATIVE 5
POINTS 10

P	0	1	2	3	4
A 0	-63273	-22449	-4809	231	231
A 1	491841	161217	25641	-7119	-2079
A 2	-1710324	-518364	-55188	36036	3276
A 3	3495996	983556	58716	-82908	8316
A 4	-4632894	-1218294	-26334	107226	-34398
A 5	4129398	1024254	-6426	-84546	49014
A 6	-2475396	-584892	14364	42084	-36036
A 7	961884	218484	-7812	-13356	14364
A 8	-219681	-48321	2079	2583	-2961
A 9	22449	4809	-231	-231	273
ERROR	-7.4219-002	-1.2355-002	8.9699-004	2.6042-004	-3.7616-004

DERIVATIVE 5
POINTS 10

P	5	6	7	8	9
A 0	-273	231	231	-4809	-22449
A 1	2961	-2583	-2079	48321	219681
A 2	-14364	13356	7812	-218484	-961884
A 3	36036	-42084	-14364	584892	2475396
A 4	-49014	84546	6426	-1024254	-4129398
A 5	34398	-107226	26334	1218294	4632894
A 6	-8316	82908	-58716	-983556	-3495996
A 7	-3276	-36036	55188	518364	1710324
A 8	2079	7119	-25641	-161217	-491841
A 9	-231	-231	4809	22449	63273
ERROR	3.7616-004	-2.6042-004	-8.9699-004	1.2355-002	7.4219-002

DERIVATIVE 6
POINTS 10

P	0	1	2	3	4
A 0	9450	4536	1638	252	-126
A 1	-77616	-35910	-11844	-882	1512
A 2	283752	126504	37800	-504	-6552
A 3	-606312	-260568	-70056	7560	14616
A 4	834876	346248	83412	-17136	-18900
A 5	-768600	-308196	-66528	19908	14616
A 6	473256	183960	35784	-13608	-6552
A 7	-187992	-71064	-12600	5544	1512
A 8	43722	16128	2646	-1260	-126
A 9	-4536	-1638	-252	126	0
ERROR	1.7436-002	4.9363-003	4.2245-004	-2.7199-004	7.5231-005

DERIVATIVE 6
POINTS 10

P	5	6	7	8	9
A 0	0	126	-252	-1638	-4536
A 1	-126	-1260	2646	16128	43722
A 2	1512	5544	-12600	-71064	-187992
A 3	-6552	-13608	35784	183960	473256
A 4	14616	19908	-66528	-308196	-768600
A 5	-18900	-17136	83412	346248	834876
A 6	14616	7560	-70056	-260568	-606312
A 7	-6552	-504	37800	126504	283752
A 8	1512	-882	-11844	-35910	-77616
A 9	-126	252	1638	4536	9450
ERROR	7.5231-005	-2.7199-004	4.2245-004	4.9363-003	1.7436-002

DERIVATIVE 5
POINTS 11

P	0	1	2	3
A 0	-902055	-269325	-44835	3255
A 1	7611660	2060520	223860	-80640
A 2	-29222865	-7201215	-405405	402885
A 3	67278960	15215760	196560	-942480
A 4	-102887190	-21598290	420210	1270710
A 5	109163880	21540960	-884520	-1083600
A 6	-81312210	-15264270	827190	619290
A 7	41937840	7565040	-468720	-246960
A 8	-14316435	-2500785	167265	68355
A 9	2917740	496440	-34860	-11760
A10	-269325	-44835	3255	945
ERROR	8.5601-002	1.1383-002	-9.7277-004	-7.5783-005

DERIVATIVE 5
POINTS 11

P	4	5	6	7
A 0	945	-1365	1365	-945
A 1	-7140	15960	-16380	11760
A 2	-28665	-82215	91035	-68355
A 3	246960	196560	-307440	246960
A 4	-630630	-203490	647010	-619290
A 5	834120	0	-834120	1083600
A 6	-647010	203490	630630	-1270710
A 7	307440	-196560	-246960	942480
A 8	-91035	82215	28665	-402885
A 9	16380	-15960	7140	80640
A10	-1365	1365	-945	-3255
ERROR	1.8463-004	-1.9152-004	1.8463-004	-7.5783-005

DERIVATIVE 5
POINTS 11

P	8	9	10
A 0	-3255	44835	269325
A 1	34860	-496440	-2917740
A 2	-167265	2500785	14316435
A 3	468720	-7565040	-41937840
A 4	-827190	15264270	81312210
A 5	884520	-21540960	-109163880
A 6	-420210	21598290	102887190
A 7	-196560	-15215760	-67278960
A 8	405405	7201215	29222865
A 9	-223860	-2060520	-7611660
A10	44835	269325	902055
ERROR	-9.7277-004	1.1383-002	8.5601-002

DERIVATIVE 6
POINTS 11

P	0	1	2	3
A 0	157773	63273	17913	1533
A 1	-1408890	-538230	-133770	1050
A 2	5684805	2071125	446985	-49455
A 3	-13655880	-4755240	-884520	194040
A 4	21636090	7224210	1156050	-378630
A 5	-23630796	-7596036	-1051596	447804
A 6	18019890	5601330	679770	-343350
A 7	-9472680	-2860200	-309960	173880
A 8	3284505	967365	95445	-57015
A 9	-678090	-195510	-17850	11130
A10	63273	17913	1533	-987
ERROR	-2.2598-002	-5.1620-003	-2.2569-004	1.9676-004

DERIVATIVE 6
POINTS 11

P	4	5	6	7
A 0	-987	273	273	-987
A 1	12390	-3990	-2730	11130
A 2	-53235	27405	11025	-57015
A 3	113400	-98280	-17640	173880
A 4	-131670	203490	-8190	-343350
A 5	77364	-257796	77364	447804
A 6	-8190	203490	-131670	-378630
A 7	-17640	-98280	113400	194040
A 8	11025	27405	-53235	-49455
A 9	-2730	-3990	12390	1050
A10	273	273	-987	1533
ERROR	-7.5231-005	*-1.5960-005	7.5231-005	-1.9676-004

DERIVATIVE 6
POINTS 11

P	8	9	10
A 0	1533	17913	63273
A 1	-17850	-195510	-678090
A 2	95445	967365	3284505
A 3	-309960	-2860200	-9472680
A 4	679770	5601330	18019890
A 5	-1051596	-7596036	-23630796
A 6	1156050	7224210	21636090
A 7	-884520	-4755240	-13655880
A 8	446985	2071125	5684805
A 9	-133770	-538230	-1408890
A10	17913	63273	157773
ERROR	2.2569-004	5.1620-003	2.2598-002