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Iterated Deferred Corrections for Nonlinear Operator Equations*

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Introduction

In [4] the author developed the method of iterated deferred corrections (IDC) for the approximate solution of nonlinear operator equations in Banach spaces by means of discretization algorithms. That method gave a way of increasing the order of convergence (with respect to h, the discretization "step") of the approximations.

The theory was based on a theorem by STETTER [8] that under certain conditions established the existence of asymptotic expansions (in powers of h) for the global discretization error. Of capital importance for the feasibility of the method was the existence of certain operators S_k which had to approximate segments of the local truncation error.

In this paper we intend to show that if the hypothesis of local stability is replaced by that of uniform stability then some of the other hypotheses in Stetter's theorem are unnecessary. We also show that the IDC method can still be defined under much weaker conditions on the operators S_k. To achieve the first objective we prove in Theorem 2.1 a nonlinear version of a classical theorem which says that:

"If the continuous problem has a solution then consistency and uniform stability of the discretization imply existence and (discrete) convergence of the approximate solution to the exact solution."

AUBIN [2], BROWDER [3], and PETRYSHYN [7] obtain similar results on a fairly different setting of the problem.

In §3 we prove our main result. This is contained in Theorem 3.2; the improvement upon our previous version ([4], Theorem 4.1) is that now the construction of the operators S_k is essentially reduced to a problem of interpolation, while before it was tied up with the result of the $(k-1)$-correction, thus making it problem dependent.

In a companion paper [6] we shall present concrete applications of the theory to nonlinear boundary value problems for ordinary differential equations. Both two-point and periodic boundary conditions will be considered and several numerical examples will be given.

§ 1. Basic Results and Definitions

Let us consider the nonlinear operator equation

$$F(x) = 0, \tag{1.1}$$

where the continuous operator F maps a subset $(\mathscr{D}(F))$ of the Banach space D into the Banach space E. We shall assume that (1.1) has a unique solution $x^* \in \mathscr{D}(F)$.

We are interested in the approximate solution of (1.1) by means of discretization algorithms. A *discretization algorithm* is a one parameter family of quintuplets $\Omega = \{\Phi_h, D_h, E_h, \varphi_h, \varphi_h^0\}$, $0 < h \leqq h_0$, where Φ_h is a continuous operator (generally nonlinear) mapping a subset $(\mathscr{D}(\Phi_h))$ of D_h into E_h, where both are finite dimensional B-spaces, and φ_h, φ_h^0 are linear bounded mappings $\varphi_h: D \to D_h$, $\varphi_h^0: E \to E_h$. The mappings φ_h, φ_h^0 are sometimes called *space discretizations*.

We require of the families $\{D_h\}$, $\{E_h\}$ that they be directed sets (ordered by inclusion: $D_{h_1} < D_{h_2}$ iff D_{h_1} is isomorphic and isometric to a subspace of D_{h_2}), i.e.: for each pair D_{h_1}, D_{h_2} there exists D_{h_3} such that $D_{h_1} < D_{h_3}$, $D_{h_2} < D_{h_3}$. We finally require that if $D_{h_1} < D_{h_2}$ then $h_1 \geqq h_2$.

With respect to the discretization mappings φ_h we ask that their ranges $\mathscr{R}(\varphi_h) \supset \mathscr{D}(\Phi_h)$, and that $\|\varphi_h\| \leqq K$, where K is a constant independent of h. Some of these hypotheses are unnecessary when the D_h, E_h are subspaces of D, E respectively, which is the setting of [3, 7].

We also assume that for each $x \in \mathscr{D}(F)$ the operators F, Φ_h are related by the asymptotic expansion

$$\Phi_h(\varphi_h x) = \varphi_h^0 \left\{ F(x) + \sum_{j=1}^{N} F_j(x)\, h^{pj} \right\} + O(h^{p\cdot(N+1)}) \tag{1.2}$$

where the operators F_j are given independently of h, and $p > 0$.

The property $\|(\Phi_h \varphi_h - \varphi_h^0 F)\, x\| \to 0$, $h \to 0$, is usually referred to as the *consistency* of the discretization.

Condition (1.2) with $N = 1$ shall be called *consistency of order p* of the discretization Φ_h with respect to F.

We will say that Φ_h is *uniformly stable* on a family of sets $\{\mathscr{A}_h\}$, $\mathscr{A}_h \subset D_h$, $0 < h \leqq h_0$, if there exists a positive constant c, independent of h such that for every V_h, $U_h \in \mathscr{A}_h$

$$\|\Phi_h(V_h) - \Phi_h(U_h)\| \geqq c\, \|V_h - U_h\|. \tag{1.3}$$

If $D_h \equiv E_h$ are infinite dimensional complex Hilbert spaces then (1.3) is implied by the *complex monotonicity condition* (cf. Zarantonello [10])

$$|(\Phi_h(V_h) - \Phi_h(U_h), V_h - U_h)| \geqq c \|V_h - U_h\|^2, \tag{1.4}$$

and the conclusion of Theorem 2.1 below still holds in this infinite dimensional case (Zarantonello [10], Theorem 1).

If Φ_h is Fréchet differentiable and its Fréchet derivative Φ_h' is uniformly Hölder continuous at V_h^*, i.e. for $\|V_h - V_h^*\| \leqq s$, $\|\Phi_h'(V_h) - \Phi_h'(V_h^*)\| \leqq L \|V_h - V_h^*\|^\alpha$, $0 < \alpha \leqq 1$, where L is independent of h, then (1.3) is implied locally by the usual condition of stability for linear operators

$$\|[\Phi_h'(V_h^*)]^{-1}\| \leqq S, \tag{1.5}$$

where S is independent of h (cf. Stetter [9], Theorem 1).

A superscript in parentheses following the name of an operator: $F^{(k)}$, will always indicate its k-th Fréchet derivative (F' is the first derivative as usual). The arguments for the multilinear operators $F^{(k)}(x)$ will sometimes be explicitly displayed as in $F^{(k)}(x)$ $(e_1, \ldots, e_k)$, and other times will not, depending on the needs of the moment. For instance: $(\varphi_h e_1, \ldots, \varphi_h e_k)$ will be shortened to $\varphi_h e$, and so on.

§ 2. Existence of Discrete Solutions and Asymptotic Expansions

We can now prove the following theorem.

Theorem 2.1. *Let x^* be the unique solution of* (1.1), *and let Ω be a consistent, uniformly stable discretization of F on the spheres $B(\varphi_h x^*, \varrho)$, where ϱ is independent of h. Then, for $0 < h \leqq h_0$, the equation*

$$\Phi_h(V) = 0 \tag{2.1}$$

has a unique solution $U(h)$ on $B(\varphi_h x^*, \varrho)$, and furthermore

$$\|U(h) - \varphi_h x^*\| \to 0 \quad as \quad h \to 0. \tag{2.2}$$

Property (1.8) will be called the *discrete convergence* of the approximate solutions $U(h)$.

Proof. By consistency we know that

$$\|\Phi_h(\varphi_h v) - \varphi_h^0 F(v)\| \to 0, \qquad h \to 0.$$

Therefore, by using x^* instead of v we obtain

$$\|\Phi_h(\varphi_h x^*)\| \to 0, \qquad h \to 0. \tag{2.3}$$

On the other hand, the stability says that the mapping Φ_h is a homeomorphism between $B(\varphi_h x^*, \varrho)$ and its image, R_h. Calling $y_h^* = \Phi_h(\varphi_h x^*)$, we have obviously $y_h^* \in R_h$, and $\|y_h^*\| \to 0$, as $h \to 0$.

By Brouwer's Domain Invariance Theorem (cf. ALEKSANDROV [*1*]) the interior of B is mapped onto the interior of R_h, and the boundary onto the boundary. Let V be such that $\|V - \varphi_h x^*\| = \varrho$. We know that $\Phi_h(V)$ belongs to the boundary of R_h, and by stability

$$\|\Phi_h(V) - y_h^*\| \geqq c\,\varrho.$$

Therefore, the distance between y_h^* and the boundary of R_h is $\geqq c\,\varrho$, and the sphere $B(y_h^*, c\,\varrho)$ is fully contained in R_h. Taking now h_0 such that $\|y_h^*\| < c\,\varrho$ it follows that $0 \in B(y_h^*, c\,\varrho) \subset R_h$ for $0 < h \leqq h_0$, thus there exists a unique solution (in $B(\varphi_h x^*, \varrho)$) of (2.1). Let us call this solution $U(h)$. By stability

$$\|U(h) - \varphi_h x^*\| \leqq c\|y_h^*\| \to 0, \quad \text{as} \quad h \to 0,$$

and the theorem is proved.

Remarks. Observe that the consistency has only been used at x^*. If Φ_h is consistent of order p it follows that $\|y_p^*\| = O(h^p)$ and consequently that

$$\|U(h) - \varphi_h x^*\| = O(h^p).$$

The vector $U(h) - \varphi_h x^*$ is sometimes called the *global discretization error* (g.d.e.). In [8], STETTER proved a theorem about the existence of asymptotic expansions in powers of h for the g.d.e. We shall state here a modified version of that theorem. Its proof can be found in all detail in [5], and we do not reproduce it here since it is very close to that of STETTER. Besides some minor differences in the setting of the problem and some generalization in the exponents that are allowed in the asymptotic expansions, the main contribution has already been included in Theorem 2.1 where it was shown that existence and discrete convergence of the approximate solutions $U(h)$ is a consequence of the uniform stability.

Theorem 2.2. *Let F and Ω be as before, and let x^* be the unique solution of $F(x)=0$ in $\mathscr{D}(F)$. Let F and Φ_h be $M+1$ times continuously Fréchet differentiable $(M \geqq 1)$. Let us also assume that:*

(i) $[F'(x^*)]^{-1}$ *exists;*

(ii) *There exists asymptotic expansion relating Φ_h, F and their Fréchet derivatives up to the M-th order:*

$$\Phi_h^{(j)}(\varphi_h x^*)(\varphi_h e_1, \ldots, \varphi_h e_j) = \varphi_h^0 \{F^{(j)}(x^*)(e_1, \ldots, e_j) + \sum_{\nu=1}^{M} F_\nu^{(j)}(x^*)(e_1, \ldots, e_j) h^{p\nu}\} + O(h^{p\cdot(M+1)}) \quad (j=0, \ldots, M), \tag{2.4}$$

where the operators $F_\nu^{(j)}(x^*)$ map D^j into E, $F_\nu^{(0)} = F_\nu$. Also the $\Phi_h^{(j)}(\varphi_h x^*)$ are uniformly bounded with respect to h. (Hypothesis (ii) for $j \geqq 1$ was omitted in STETTER'S Theorem.)

(iii) $[\Phi_h'(\varphi_h x^*)]^{-1}$ *exists and satisfies* (1.5).

Let $U(h)$ be the unique solutions of the equations $\Phi_h(V)=0$ (in $B(\varphi_h x^*, \varrho)$, $h \leqq h_0$) which exist and are convergent of order p, i.e. $\|U(h) - \varphi_h x^*\| = O(h^p)$.

Then the global discretization error $e(h) = U(h) - \varphi_h x^*$ has an asymptotic expansion of the form

$$e(h) = \varphi_h \sum_{\nu=1}^{M} e_\nu h^{p\nu} + O(h^{p\cdot(M+1)}), \tag{2.5}$$

where the $e_\nu \in D$, and are independent of h. Moreover, the e_ν are solutions of the linear equations

$$F'(x^*) e_\nu = b_\nu \quad (\nu = 1, \ldots, M),$$

where the b_ν can be recursively constructed.

Remark. It is not necessary that the $F_\nu^{(j)}$ be the Fréchet derivatives of the F_ν. The theorem is true if the $F_\nu^{(j)}(x^*)$ are replaced by given j-linear operators, not necessarily bounded.

§ 3. Iterated Deferred Corrections (IDC)

In [4, §4] we developed the method of iterated deferred corrections. The critical assumption there was the existence of sufficiently accurate discretization operators for the local truncation error, i.e.: provided that $U^{(k)}(h)$ was convergent of order $p\cdot(k+1)$, we needed operators S_k such that

$$\varphi_h^0 \sum_{j=1}^{k+1} h^{pj} F_j(x^*) - S_{k+1}(U^{(k)}) = O(h^{p\cdot(k+2)}). \tag{3.1}$$

With this we could easily prove that the solution $U^{(k+1)}(h)$ to the problem $\Phi_h(V) = S_{k+1}(U^{(k)})$ satisfied $U^{(k+1)}(h) - \varphi_h x^* = O(h^{p\cdot(k+2)})$, and therefore we could increase the order of accuracy of our approximate solution. The effective construction of the operators S_k was left to the reader, with the exception of a few examples that we worked out at the end of the paper.

Now we wish to present a more complete version of that result in which the existence of the operators S_k is based on much weaker assumptions. Since the whole procedure is of a recursive nature all our proofs are by induction and we will state the results at the k-th step.

Let us consider then problem (1.1) with the discretization Ω satisfying the hypotheses of Theorem 2.2, with $M = 2N$ (the restriction to even M is just a matter of convenience in the proofs). Let $U^{(0)}$ equal $U(h)$, the solution of (2.1).

Induction hypothesis: For a given k, $1 \leqq k \leqq N$, there exists $U^{(k-1)}(h)$ such that

$$U^{(k-1)}(h) - \varphi_h x^* = \varphi_h \sum_{\varrho=k}^{M-k+1} e_{k-1,\varrho} h^{p\varrho} + O(h^{p\cdot(M-k+2)}), \tag{3.2}$$

where the $e_{k-1,\varrho} \in D$ are independent of h.

That this condition is fulfilled for $k=1$ follows from Theorem 2.2. We set now to prove that a $U^{(k)}(h)$ exists such that (3.2) holds. Let us first of all define

$$\overline{F}_k(x) = \sum_{j=1}^{k} F_j(x) h^{pj}, \qquad N_k = \left[\frac{M-k+1}{k}\right]. \tag{3.3}$$

Lemma 3.1. *Let us assume that the conditions of Theorem 2.2 and (3.2) are fulfilled, and that $\overline{F}_k$ is $\infty-$ differentiable. Assume also that there exists a $\infty-$ differentiable operator S_k, satisfying for all j and $e \in D^j$*

$$\begin{aligned} &\|S_k^{(j)}(\varphi_h x^*)\| = O(h^{-p}), \\ &S_k^{(j)}(\varphi_h x^*)\varphi_h e = \varphi_h^0 \left\{\overline{F}_k^{(j)}(x^*) e + \sum_{\nu=k+1}^{M-k} t_{k\nu}^{(j)}(x^*) e h^{p\nu}\right\} + O(h^{p(M-k+1)}) \end{aligned} \tag{3.4}$$

with $t_{k\nu}$ being given operators independent of h. Then

$$\begin{aligned} &\Phi_h^{(j)}(\varphi_h x^*)\varphi_h e - S_k^{(j)}(U^{(k-1)})\varphi_h e \\ &\quad = \varphi_h^0 \left\{F^{(j)}(x^*) e + \sum_{\nu=k+1}^{M-k} F_{k\nu}^{(j)}(x^*) e h^{p\nu}\right\} + O(h^{p(M-k+1)}), \end{aligned} \tag{3.5}$$

where the $F_{k\nu}$ are $(M-k+1)$-times differentiable operators independent of h.

Proof. First of all we will establish a useful representation for $(U^{(k-1)} - \varphi_h x^*)^r$. By (3.2) we know that

$$\begin{aligned} (U^{(k-1)} - \varphi_h x^*)^r &= \left[\sum_{\varrho=k}^{M-k+1} \varphi_h e_{k-1,\varrho} h^{p\varrho} + O(h^{p(M-k+2)})\right]^r \\ &= \left(\sum_{\varrho=k}^{M-k+1} \varphi_h e_{k-1,\varrho} h^{p\varrho}\right)^r + O(h^{p(M-k+2)}), \end{aligned}$$

and therefore

$$(U^{(k-1)} - \varphi_h x^*)^r = \begin{cases} \sum_{\varrho=kr}^{M-k+1} g_{r\varrho}(\varphi_h e_{k-1,k}, \ldots, \varphi_h e_{k-1,\varrho-k(r-1)})\, h^{p\varrho} \\ \qquad\qquad + O(h^{p(M-k+2)}) \quad \text{for} \quad r \leqq N_k, \\ O(h^{p(M-k+2)}) \quad \text{for} \quad r > N_k, \end{cases} \tag{3.6}$$

where the $g_{r\varrho}$ are polynomial forms r-homogeneous in their variables. By the general Taylor formula and (3.3) we have for $j=1, \ldots, M-k+1$

$$S_k^{(j)}(U^{(k-1)}) - S_k^{(j)}(\varphi_h x^*) = \sum_{r=1}^{N_k} \frac{S_k^{(j+r)}(\varphi_h x^*)}{k!} (U^{(k-1)} - \varphi_h x^*)^r + O(h^{p(M-k+1)}),$$

and using (3.4) (for $S_k^{(j+r)}$) and (3.6) in this last expression we obtain

$$\begin{aligned} S_k^{(j)}(U^{(k-1)}) - S_k^{(j)}(\varphi_h x^*) = &\left\{\varphi_h^0 \sum_{r=1}^{N_k} \bar{F}_k^{(j+r)}(x^*) + \sum_{\nu=k+1}^{M-k} t_{k\nu}^{(j+r)}(x^*)\, h^{p\nu}\right\} \\ &\cdot \sum_{\varrho=kr}^{M-k+1} g_{r\varrho}(e_{k-1,k}, \ldots, e_{k-1,\varrho-k(r-1)})\, h^{p\varrho} + O(h^{p(M-k+1)}). \end{aligned}$$

Recalling the definition of $\bar{F}_k^{(j+r)}(x^*)$ and reordering the right hand side of the above expression we get

$$S_k^{(j)}(U^{(k-1)}) - S_k^{(j)}(\varphi_h x^*) = \varphi_h^0 \sum_{\nu=k+1}^{M-k} s_{k\nu}^{(j)}(x^*)\, h^{p\nu} + O(h^{p(M-k+1)})$$

where the $s_{k\nu}^{(j)}(x^*)$ are j-linear operators independent of h. Using (3.4) again (now for $S_{k\nu}^{(j)}(\varphi_h x^*)$) we obtain for $e=(e_1, \ldots, e_j) \in D^j$

$$S_k^{(j)}(U^{(k-1)})\, \varphi_h e = \varphi_h^0 \left\{\bar{F}_k^{(j)}(x^*)\, e + \sum_{\nu=k+1}^{M-k} [s_{k\nu}^{(j)}(x^*) + t_{k\nu}^{(j)}(x^*)]\, e\, h^{p\nu}\right\} + O(h^{p(M-k+1)}),$$

and finally, by (2.4)

$$\begin{aligned} &[\Phi_k^{(j)}(\varphi_h x^*) - S_k^{(j)}(U^{(k-1)})]\, \varphi_h e \\ &\quad = \varphi_h^0 \left\{F^{(j)}(x^*)\, e + \sum_{\nu=k+1}^{M-k} [s_{k\nu}^{(j)}(x^*) + t_{k\nu}^{(j)}(x^*) + F_\nu^{(j)}(x^*)]\, e\, h^{p\nu}\right\} + O(h^{p(M-k+1)}). \end{aligned}$$

If we set $F_{k\nu} = s_{k\nu} + t_{k\nu} + F_\nu$ the lemma follows.

Defining now

$$U^{(-1)} = 0, \qquad S_0 \equiv 0, \tag{3.7}$$

we can establish our main result, which is an improved version of Theorem 4.1 of [4].

Theorem 3.2. *For problem* (1.1) *with the discretization Ω assume that the conditions of Theorem 2.2 are satisfied, and that there exist operators S_k ($k=0, \ldots, N$) as described in* (3.7) *and Lemma 3.1. In this case, the method of iterated deferred corrections given by*

$$\Phi_h(U^{(k)}) - S_k(U^{(k-1)}) = O(h^{p(M-k+1)}) \tag{3.8}$$

is well defined and any solution of the inequality (3.8) *satisfies*

$$U^{(k)} - \varphi_h x^* = \varphi_h \sum_{\varrho=k+1}^{M-k} e_{k\varrho} h^{p\varrho} + O(h^{p(M-k+1)}). \tag{3.9}$$

Proof. By Lemma 3.1 the discretization (3.8) is consistent of order $p(k+1)$. It is stable because $\Phi_h(V)$ is stable and $S_k(U^{(k-1)})$ is just a constant term. Therefore, Theorem 2.1 asserts that (3.8) has a solution which is convergent of order $p(k+1)$, and Theorem 2.2 can be applied in order to obtain (3.9).

In a completely similar way we can prove the following theorem.

Theorem 3.3. *For problem* (1.1) *with the discretization* Ω *assume that the conditions of Theorem 2.2 are satisfied with the exception of* (ii). *Instead of* (ii) *we assume:*

(ii_k) *There exist asymptotic expansions relating* Φ_h, F *and their Fréchet derivatives up to the M-th order:*

$$\Phi_h^{(j)}(\varphi_h x^*)(\varphi_h e_1,\ldots,\varphi_h e_j)$$
$$=\varphi_h^0\Big\{F^{(j)}(x^*)(e_1,\ldots,e_j)+\sum_{\nu=1}^{k}F_{\nu k}^{(j)}(x^*)(e_1,\ldots,e_j)\,h^{p\nu} \tag{2.4'}$$
$$+\sum_{\nu=pk+1}^{pM}F_{\nu k}^{(j)}(x^*)(e_1,\ldots,e_j)\,h^{\nu}\Big\}+O(h^{pM+1})\quad (j=0,\ldots,M;\ k=1,\ldots,[M/2]).$$

Assume also that there exists operators S_k $(k=0,\ldots,N)$ *satisfying:*

$$S_0\equiv 0,\qquad \|S_k^{(j)}(\varphi_h x^*)\|=O(h^{-p}),$$
$$S_k^{(j)}(\varphi_h x^*)\,\varphi_h e=\varphi_h^0\Big\{\overline{\overline{F}}_k^{(j)}(x^*)\,e+\sum_{\nu=pk+1}^{p(M-k)}t_{k\nu}^{(j)}(x^*)\,e\,h^{\nu}\Big\}+O(h^{p(M-k)+1}), \tag{3.4'}$$

where $\overline{\overline{F}}_k(x)=\sum_{j=1}^{k}F_{\nu k}(x)\,h^{pj}$. *In this case, the method* (3.8) *is well defined and any solution of that inequality satisfies:*

$$U^{(k)}-\varphi_h x^*=\varphi_h\sum_{\varrho=p(k+1)}^{p(M-k)}e_{k\varrho}\,h^{\varrho}+O(h^{p(M-k)+1}). \tag{3.9'}$$

Proof. It is enough to observe that with the assumption (3.4′) the same proof as that of Lemma 3.1 gives (3.5) with a right hand side containing *all* consecutive powers of h, between $p(k+1)$ and $p(M-k)$.

Remarks. From the proofs above it is clear that the differentiability requirements for $\overline{F}_k$ can be weakened by asking that they belong to C^t, for t sufficiently large. A more interesting observation is that in the important case in which the $\overline{F}_k$ are linear then it is enough to have $t=1$. In fact the hypotheses on differentiability of the $\overline{F}_k$ ($\overline{\overline{F}}_k$ in Theorem 3.3) can be dropped altogether if, as we remarked after Theorem 2.2, the $F_\nu^{(j)}(x^*)$ (or $F_{\nu k}^{(j)}(x^*)$) are replaced by given j-linear operators. Of course, in this case we will obtain in (3.5) (Lemma 3.11), instead of the $F_{k\nu}^{(j)}(x^*)$ certain j-linear operators, but this is all what is needed in Theorems 3.2 and 3.3.

By means of Lemma 3.1 we have effectively reduced the task of constructing the operators S_k with the property (3.1). In fact, now it is enough to find any $p(k+1)$-consistent discretization for $\overline{F}_k(x)$, $x\in\mathcal{D}(F)$.

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