Numer. Math. 23, 261 – 268 (1975)

F

Mesh Selection for Discrete Solution of Boundary Problems in Ordinary Differential Equations

V. Pereyra and E. G. Sewell

Received June 7, 1974

Summary. In order to use finite difference approximations with non-uniform meshes in boundary value problems, it is necessary to develop procedures for mesh selection. In this paper we extend techniques that have been used in piecewise polynomial approximation which permit the construction of equidistributing meshes. By this term we mean meshes on which the local truncation error of the method is approximately constant in some norm. Improved error estimates for methods which use equidistributing meshes are obtained.

1. Introduction

Discrete variable methods are a common technique for solving boundary value problems in ordinary differential equations [5]. Until fairly recently most algorithms proposed used only uniform meshes and we find very little in the literature about the use and choice of nonuniform meshes (see however [2, 3, 11]).

In this paper we shall be interested mainly in methods that do not use initial value or marching algorithms since in that case the results on mesh selection from initial value problems techniques carry over, at least partially. We are not saying, by any means, that shooting techniques have no mesh placement problems left to be solved. As a matter of fact, one of the most successful techniques of that type is multiple shooting, an hybrid between simple shooting and full finite differences. To our knowledge, algorithms for the optimal automatic placement of shooting nodes are still not available.

The work of Keller [5-7] has filled partially the information gap by providing an efficient, simple, and fully analized finite difference procedure for first order nonlinear systems with very general boundary conditions, which can be used with nonuniform meshes without losing any of its convergence properties.

However, very little has been added about the problem of choosing those nonuniform meshes in the way most adequate for a given problem. Besides the references we mentioned above the most outstanding recent contribution is due to Carl de Boor [1]. De Boor is concerned in his paper with the choice of the location of knots for the solution of scalar m-order nonlinear differential equations by collocation with piecewise polynomial functions.

It is the purpose of this paper to extend the ideas of de Boor to discrete variable approximations for more general boundary problems, and to obtain some new improved error estimates for methods which use appropiately chosen meshes. Since the technique is not really attached to a particular type of boundary problem or method we shall make a somewhat abstract presentation in order to emphasize the relevant facts and generality.

In Section 3 we introduce the notions of equidistributing and approximately equidistributing meshes and show how to construct this last kind. An improved error estimate for methods using equidistributing meshes is proved in Section 4. Finally in Section 5 we show some sharper results for the modified Euler method.

Based on the $O(h^2)$ method of Keller, very powerful high order methods using deferred corrections have been developed [8–10, 13]. The automatic mesh placement algorithm used in [9, 10] is based on the results of this paper and the numerical experiments performed show that the technique we are proposing here is sound and valuable.

2. Notation and Basic Results

In the spirit of [12] and other earlier work we shall consider boundary problems for ordinary differential equations symbolized by the operator equation

$$F(y) = 0, \tag{2.1}$$

where y is a m-vector function of one real variable ($m \geqq 1$) belonging to some appropriate class of smooth functions on a finite interval $[a, b]$, and satisfying some given boundary conditions. We assume that problem (2.1) has an isolated solution $y^*(t)$. A partition π of $[a, b]$ is a set of abscissas satisfying

$$a \leqq x_1 < x_2 < \cdots < x_{N+1} \leqq b. \tag{2.2}$$

The mesh sizes are defined as $h_i = x_{i+1} - x_i$; $h = \max_i h_i$. In order to assure that $1/N$ and h are equivalent asymptotic scales we shall assume that all the partitions considered satisfy

$$\frac{h}{\min_i h_i} \leqq K, \tag{2.3}$$

with K a given constant. Inequality (2.3) implies that $\frac{b-a}{N} \leqq h \leqq \frac{K(b-a)}{N}$.

A discrete variable method for finding approximate values of the solution $y^*(t)$ of (2.1) on the mesh π can be represented by an equation of the form

$$F_\pi(Y) = 0, \tag{2.4}$$

where Y is an $m x(N+1)$ dimensional vector of unknowns, and the nonlinear operator F_π maps $R^{mx(N+1)}$ into itself.

We shall use various norms in this paper. Continuous vector $L^{p,q}$ norms on $[a, b]$ are defined as

$$|||f|||_{p,q} = \left(\int_a^b \|f(t)\|_p^q\right)^{1/q},$$

and discrete $L^{p,q}$ norms for grid functions defined on π as

$$\|f\|_{p,q} = \left(\sum_{i=1}^N h_i \|f_i\|_p^q\right)^{1/q},$$

where $\|f\|_p^p = (\sum_{i=1}^m |f_i|^p)$ is the usual p-vector norm. All the results given in this paper are valid for $1 \leqq p \leqq \infty$, although they are only proved for $1 \leqq p < \infty$.

A discretization mapping Φ_π relates the space of the continuous variables $y(t)$ with that of the discrete variables Y. Observe that the discrete norms of functions $\Phi_\pi f$ are $O(h)$ approximations to the corresponding continuous norms of $f(t)$.

The local truncation error is defined as:

$$\tau = F_\pi(\Phi_\pi y^*). \tag{2.5}$$

The discrete method is consistent of order $n \geqq 1$ iff $\tau = O(h^n)$. It is well known that consistency is not sufficient for the convergence of a method. Stability is the property which together with consistency ensures the existence of solutions of (2.4) and their discrete convergence [12]. We shall say that the discrete method is p-stable if for all mesh partitions satisfying (2.3), h sufficiently small, and all mesh functions U, V, we have that

$$\|U - V\|_{p,p} \leqq c \, \|F_\pi(U) - F_\pi(V)\|_{p,p}, \tag{2.6}$$

where the constant c is independent of π.

If Y_π are solutions of (2.4) for partitions π with $h \downarrow 0$, we say that they converge discretely to $y^*(t)$ with order n iff

$$\|Y_\pi - \Phi_\pi y^*\|_{p,p} = O(h^n). \tag{2.7}$$

Applying (2.6) with $U = Y_\pi$ and $V = \Phi_\pi y^*$, we see that a stable, n-consistent method is convergent of order n.

3. Mesh Selection

In this Section we shall study a mesh selection procedure for any stable method whose local truncation error satisfies:

$$\tau_i = h_i^n T(x_i) + O(h^{n+1}), \qquad i = 1, \ldots, N, \tag{3.1}$$

where the m-dimensional vector function $T(x)$ is independent of the partition π. Our aim will be to find a mesh, with as few points as possible, and such that for a prescribed tolerance

$$\|\tau\|_{p,p} \leqq \text{tol}. \tag{3.2}$$

Ideally we would like to consider an expression like (3.1) for the global truncation error, but in general such an expression, giving the local dependency with respect to the mesh size, will not be available for boundary value problems. Nevertheless, once we know that $\|\tau\|_{p,p} \leqq \text{tol}$, then the p-stability will provide the global error estimate:

$$\|e\|_{p,p} \leqq c \ \text{tol}. \tag{3.3}$$

With this purpose we extend de Boor's ideas [1] to the present case in order to choose the mesh giving the least error for a given number of nodes. Later on we shall see how to choose the number of nodes so that the local error satisfies (3.2).

De Boor's prescription calls for the equidistribution of the norm $\|\tau\|_{p,p}$, i.e.

$$h_i \|\tau_i\|_p^p = \text{constant} = \tilde{E}, \quad i = 1, \ldots, N \tag{3.4}$$

($p = \infty$, and τ a numerical function is the case considered in [1]). A mesh such that (3.4) holds is called *equidistributing*.

Obviously

$$\tilde{E} = \frac{\|\tau\|_{p,p}^p}{N}. \tag{3.5}$$

Expression (3.5) for $\tilde{E}$ depends upon the partition π. In order to eliminate that dependence we introduce the notion of a mesh being approximately equidistributing.

Let $\|T(t)\|_p \leqq M_1$ for $a \leqq t \leqq b$, and $\varepsilon = M_1/K^{1/\sigma}$, with $\sigma = p/(np+1)$ (see (2.3) for the definition of K).

Let $g(t) = \max\left(\|T(t)\|_p, \varepsilon\right)$, and $\gamma_i = h_i^n g(x_i)$.

A mesh π is *approximately equidistributing* if $h_i \gamma_i^p = E\left(1 + O(h)\right)$, where E is a positive constant.

Observe that

$$\gamma_i = \max\left(\|\tau_i\|_p, h_i^n \varepsilon\right) \quad \left(1 + O(h)\right). \tag{3.6}$$

If $\|\tau_i\|_p$ is everywhere greater than $h_i^n \varepsilon$ then the equidistributing condition simply reads $h_i \|\tau_i\|_p^p = E\left(1 + O(h)\right)$. The introduction of the γ_i is necessary since, otherwise small values of $\|\tau_i\|$ could force inconveniently large steps h_i.

Lemma 3.1. *With the notation and hypothesis above, if the mesh π is such that $\int_a^b g(t)^\sigma dt$ is equidistributed on π* (i.e. $\int_{x_i}^{x_{i+1}} g(t)^\sigma dt = \text{constant}$), *then*

$$\|\tau\|_{p,p} \leqq N^{-n}\left(\int_a^b g(t)^\sigma dt\right)^{1/\sigma} \quad \left(1 + O(h)\right), \tag{3.7}$$

and π is approximately equidistributing with

$$E = \left[\frac{\int_a^b g(t)^\sigma dt}{N}\right]^{np+1}. \tag{3.8}$$

Proof. Since $\varepsilon \leqq g(t) \leqq M_1$, we have that

$$h_i \varepsilon^\sigma \leqq \int_{x_i}^{x_{i+1}} g(t)^\sigma dt \leqq h_i M_1^\sigma$$

or

$$h_i \varepsilon^\sigma \leqq N^{-1} \int_a^b g(t)^\sigma dt \leqq h_i M_1^\sigma,$$

which implies that $h/h_{\min} \leqq K$. Thus π satisfies the basic requirement (2.3).

On the other hand, since $g(t) \geqq \varepsilon$, we have:

$$\begin{aligned} h_i \gamma_i^p &= \left(h_i g(x_i)^\sigma\right)^{np+1} \\ &= \left(\int_{x_i}^{x_{i+1}} g(t)^\sigma dt\right)^{np+1} \quad \left(1 + O(h)\right). \end{aligned} \tag{3.9}$$

Since $\int_a^b g(t)^\sigma dt$ is equidistributed on π by hypothesis, we see that $h_i\gamma_i^p = cte^{np+1}(1+O(h))$; but $cte = \dfrac{\int_a^b g(t)^\sigma dt}{N}$, and π is approximately equidistributing with E given in (3.8).

Rewritting (3.9) according to this observation and adding up, we obtain:

$$\sum_{i=1}^{N} h_i\gamma_i^p = N^{-np}\left(\int_a^b g(t)^\sigma dt\right)^{np+1} \quad (1+O(h))$$

or

$$\left(\sum_{i=1}^{N} h_i\gamma_i^p\right)^{1/p} = N^{-n}\left(\int_a^b g(t)^\sigma dt\right)^{1/\sigma} \quad (1+O(h)).$$

But $\|\tau_i\|_p^p \leq \gamma_i^p(1+O(h))$, and therefore

$$\|\tau\|_{p,p} \leq \left(\sum_{i=1}^{N} h_i\gamma_i^p\right)^{1/p} \quad (1+O(h))$$

from where it follows (3.7). ‖

All our previous developments presupposed that the number of points in the mesh had been given a priori. We can, however, use these results in a much more interesting fashion in order to fulfill (3.2) and predict the necessary N for a given tol. By (3.7) we see that, assuming equilibration, we can approximately satisfy (3.2) by choosing N so that

$$N^{-n}\left(\int_a^b g(t)^\sigma dt\right)^{1/\sigma} \leq \text{tol}.$$

Introducing $\|\Phi_\pi g\|_\sigma \equiv \left(\sum_{i=1}^N h_i g(x_i)^\sigma\right)^{1/\sigma}$, we have an approximate formulation of this inequality:

(3.10) $$N^{-n}\|\Phi_\pi g\|_\sigma(1+O(h)) \leq \text{tol}.$$

From (3.8) and (3.10) (disregarding the $O(h)$ term) we eliminate N and derive a relationship between the level of equilibration E and the desired precision ε:

(3.11) $$E \leq (\text{tol}\,\|\Phi_\pi g\|_\sigma^{\sigma n-1})^{p/\sigma n}.$$

By means of an iterative procedure, adding or removing points as necessary to equilibrate to level E, an approximately equilibrating mesh is produced such that (3.2) is fulfilled.

An algorithm based on these results has been implemented as a basic component in an adaptive finite difference procedure for solving two-point problems for nonlinear first order systems which can handle successfully boundary layers and other nonstandard difficulties [9, 10].

Burchard [15] has given a proof similar to that of Lemma 3.1, for the problem of function approximation by piecewise polynomials. See also [4, 14].

4. Error Estimates for Equidistributing Meshes

In this Section we assume that the method in consideration is stable in norm p, and that the mesh is equidistributing so that (3.7) holds. Therefore the following estimate for the global error e is evident from (2.6) and (3.7)

$$\|e\|_{p,p} = cN^{-n} |||g|||_\sigma + O(h^{n+1}), \tag{4.1}$$

with $|||g|||_\sigma = (\int_a^b |g(t)|^\sigma dt)^{1/\sigma}$.

Since $|||\cdot|||_\sigma$ is not a norm ($\sigma < 1$), it is desirable to have an estimate in $|||\cdot|||_p$. This estimate is provided by the next Lemma.

Lemma 4.1. *Let $p \geqq 1$, let $f(x)$ be a scalar function satisfying $|||f(x)|||_p < \infty$, and let $0 < \sigma < p$. Let $M_L = \{x / |f(x)| \geqq L\}$ with L chosen so that*

$$\int_{M_L} |f|^\sigma dx = 1/2 \int_a^b |f|^\sigma dx = 1/2\, |||f|||_\sigma^\sigma.$$

Then

$$|||f|||_\sigma \leqq 2^{1/\sigma} [\mu(M_L)]^{(p-\sigma)/\sigma p} |||f|||_p, \tag{4.2}$$

where $\mu(M_L)$ is the measure of the set M_L.

Proof. Since $p/\sigma > 1$, let q be such that $(p/\sigma)^{-1} + q^{-1} = 1$ (σ could be < 1!). By Holder's inequality we have

$$|||f|||_\sigma^\sigma = \int_a^b |f|^\sigma \cdot 1\, dx \leqq \left[\int_a^b (|f|^\sigma)^{p/\sigma} dx\right]^{p/\sigma} \left[\int_a^b 1^q dx\right]^{1/p},$$

or

$$|||f|||_\sigma \leqq |||f|||_p (b-a)^{(\sigma^{-1} - p^{-1})}. \tag{4.3}$$

But from one of the hypotheses $|||f|||_\sigma = 2^{1/\sigma} (\int_{M_L} |f|^\sigma dx)^{1/\sigma}$, and if we apply (4.3) to this last expression we get:

$$|||f|||_\sigma \leqq 2^{1/\sigma} \left(\int_{M_L} |f|^p\right)^{1/p} [\mu(M_L)]^{(\sigma^{-1} - p^{-1})}.$$

From this last inequality follows (4.2). ‖

Combining (4.1) and (4.2) we obtain:

Theorem 4.2. *Let F be a p-stable method whose local truncation error satisfies (3.1), and let the mesh π have $N+1$ points and be equidistributing. Then, its global truncation error satisfies*

$$\|e\|_{p,p} = \frac{c\, 2^{1/\sigma} \mu(M_L)^{(p-\sigma)/\sigma p}}{N^n} |||g|||_p + O(h^{n+1}). \quad \| \tag{4.4}$$

Observe that $\mu(M_L)$ will be small if $T(x)$ has sharp peaks and compare with (4.1).

5. A Sharper Result for a Particular Finite Difference Scheme

We consider the first order system

$$F(y) \equiv y'(x) - F(x, y(x)) = 0 \quad a \leqq x \leqq b \quad F(x, y), y(x) \in R^n \tag{5.1}$$

and use the modified Euler scheme [5, 13]

$$[F_\pi(Y)]_j \equiv \frac{1}{h_j}(Y_{j+1} - Y_j) - \frac{1}{2}[f(x_j, Y_j) + f(x_{j+1}, Y_{j+1})] = 0 \tag{5.2}$$

to approximate an isolated solution y^* of (5.1).

The truncation error for (5.2), ignoring boundary conditions, can be written as

$$\tau_j = [F_\pi(\Phi_\pi y^*)]_j = -\frac{h_j^2}{12} y^{*(3)}(x_j) + O(h^2)$$

or as

$$\tau_j = \frac{-1}{2h_j} \int_{x_j}^{x_{j+1}} y^{*(3)}(s)(x_{j+1} - s)(s - x_j)\, ds.$$

Then

$$\|\tau_j\|_\infty \leqq \frac{1}{2h_j} \int_{x_j}^{x_{j+1}} \|y^{*(3)} s)\|_\infty (x_{j+1} - s)(s - x_j)\, ds.$$

We shall need the following inequality taken from [4], whose proof is a simple exercise and will not be repeated here.

Lemma 5.1. *For any $n \geqq 1$, if $g(x)$ is nonnegative and nondecreasing in (α, β) then*

$$\int_\alpha^\beta (\beta - s)^{n-1} g(s)\, ds \leqq \frac{1}{n} \left[\int_\alpha^\beta g(s)^{1/n}\, ds\right]^n.$$

Thus if $\|y^{*(3)}(s)\|_\infty$ is nondecreasing in (x_j, x_{j+1}) we have

$$\begin{aligned} \|\tau_j\|_\infty &\leqq \frac{1}{2} \int_{x_j}^{x_{j+1}} (x_{j+1} - s)\, \|y^{*(3)}(s)\|_\infty\, ds \\ &\leqq \frac{1}{4} \left[\int_{x_j}^{x_{j+1}} \|y^{*(3)}(s)\|_\infty^{1/2}\, ds\right]^2. \end{aligned} \tag{5.3}$$

By symmetry, we clearly still have (5.3) if $\|y^{*(3)}(s)\|_\infty$ is nonincreasing in (x_j, x_{j+1}).

Suppose π_0 is a partition with knots $a = x_1 < x_2 < \cdots < x_{N_0+1} = b$ such that $\|y^{*(3)}(x)\|_\infty$ is monotone in each (x_j, x_{j+1}), and suppose π_N is the partition which distributes $\int \|y^{*(3)}(s)\|_\infty^{1/2} ds$ equally over its N intervals (x_i, x_{i+1}), $i = 1, \ldots, N$. Then $\pi = \pi_N \cup \pi_0$ has at most $N + N_0 - 1$ intervals (x_i, x_{i+1}) and in each we have, from (5.3)

$$\|\tau_j\|_\infty \leqq \frac{1}{4} \left[\frac{1}{N} \int_a^b \|y^{*(3)}(s)\|_\infty^{1/2}\, ds\right]^2$$

and

$$\max_{1\leq j\leq N} \|\tau_j\|_\infty \leqq \frac{3}{N^2} \||-\frac{1}{12} y^{*(3)}\||_{\infty,1/2} \tag{5.4}$$

which is essentially (3.7), with $p=\infty$, $n=2$, $\sigma=\frac{1}{2}$, and $T(x)=-\frac{1}{12}y^{*(3)}(x)$.

It should be noted that we do not require $M_1 \geqq \|y^{*(3)}(t)\|_\infty \geqq M_2 > 0$. In fact, $\|y^{*(3)}(t)\|_\infty$ can have singularities at the knots of π_0.

Acknowledgement. The authors are grateful to M. Lentini for her many comments and critical reading of this paper, and to Carl de Boor and H. B. Keller for interesting conversations on the subject.

References

1. De Boor, C.: Good approximation by splines with variable knots. II. Conference on the Numerical Solution of Differential Equations. Lecture Notes in Mathematics **363**, 12–20. Berlin-Heidelberg-New York: Springer 1973
2. Brown, R. R.: Numerical solution of boundary value problems using nonuniform grids. J. SIAM **10**, 475–495 (1962)
3. Denny, V. E., Landis, R. B.: A new method for solving two-point boundary value problems using optimal node destribution. J. Comp. Phys. **9**, 120–137 (1972)
4. Dodson, D. S.: Optimal order approximation by polynomial spline functions. Ph. D. Thesis, Purdue Univ., Lafayette, Ind. (1972)
5. Keller, H. B.: Numerical methods for two-point boundary-value problems. Waltham, Mass.: Blaisdell 1968
6. Keller, H. B.: Accurate difference methods for linear ordinary differential systems subject to linear constraints. SIAM J. Numer. Anal. **6**, 8–30 (1969)
7. Keller, H. B.: Some new results on difference methods for two point boundary value problems. Actas, Seminario Métodos Numéricos Modernos I, pp. 63–82. Depto. de Comp., Fac. Ciencias, Univ. Central de Venezuela, Caracas (1974)
8. Lentini, M., Pereyra, V.: A variable order finite difference method for nonlinear multipoint boundary value problems. Pub. 73–06, Depto. de Comp., Fac. Ciencias, Univ. Central de Venezuela, Caracas (1973). To appear in Math. Comp. Oct. 1974
9. Lentini, M., Pereyra, V.: Boundary problem solvers for first order systems based on deferred corrections. Proceedings NSF Symposium, Numerical Solutions of Boundary Problems. Univ. of Maryland Baltimore City. (1974)
10. Lentini, M., Pereyra, V.: An adaptive solver for nonlinear two-point first order system problems with mild boundary layers. In preparation
11. Pearson, C. E.: On a differential equation of boundary layer type. J. Math. Phys. **47**, 134–154 (1968)
12. Pereyra, V.: Iterated deferred corrections for nonlinear operator equations. Numer. Math. **10**, 316–323 (1967)
13. Pereyra, V.: Variable order variable step finite difference methods for nonlinear boundary value problems. Conference on the Numerical Solution of Differential Equations. Lecture Notes in Mathematics **363**, 118–133. Berlin: Springer 1973
14. Sewell, E. G.: Automatic generation of triangulations for piecewise polynomial approximation. Ph. D. Thesis, Purdue Univ., Lafayette, Ind. (1972)
15. Burchard, H. G.: Splines (with optimal knots) are better. J. App. Anal. **3**, 309–319 (1974)

V. Pereyra
Depto. de Computación
Apartado 59002
Caracas

E. G. Sewell
Depto. de Matemáticas
Univ. Simón Bolívar
Caracas