Numerical Solutions of Nonlinear Differential Equations

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where μ and K are known constants. The period of oscillations associated with Eq. (1) was effectively found by Knowles in an integral form. The purpose of this study is to test the accuracy of an approximate method of solution of Eq. (1) so that the method can be applied to solution of related more complex problems (such as e.g. coupled radial and longitudinal oscillations) not tractable by (specific) Knowles approach. The approximation consists in representing the solution to (1) by an expression similar to that obtained when the wall thickness is small,

$$x(t) = \frac{1}{C}\,[1 + D\cos \omega t]^{1/2} \tag{2}$$

where C, D and ω are unknown constants determined successively from: (a) the initial conditions, (b) Galerkin weighted residual procedure and (c) equation of conservation of energy.

In contrast to the rigorous solution, the numerical work involved in the evaluation of the approximate solution is trivial.

Both solutions are in perfect agreement in two limit cases: (a) when the oscillations are small and the wall thickness arbitrary, $|x-1| \ll 1$, (b) when the oscillations are arbitrary and the wall thickness small, $\mu \ll 1$. In the intermediate range the difference between the two solutions is satisfactory (less than 1.5%) and may be partially due to an unavoidable inaccuracy in computation of rigorous data (using IBM 1620 II-D with accuracy up to 16 digits). Both solutions disclose the same hard-spring behavior of oscillations.

EXTRAPOLATION TO THE LIMIT FOR FUNCTIONAL EQUATIONS

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Let $F(u) = 0$ (1) be a non-linear functional equation. $F: D \to E$, D and E Banach spaces. Let $\Phi_h(V)$ be a family of discretization algorithms for solving (1). $\Phi_h: D_h \to E_h$. Let Δ_h be linear bounded operators, $\Delta_h: D \to D_h$. Assume that u^* is the unique solution of (1) and $U(h)$ that of $\Phi_h(V) = 0$. Assume that, for a <u>fixed</u> $N \geq 1$

$$e(h) = \Delta_h u^* - U(h) = \Delta_h \sum_{j=1}^{N} h^{p_j} e_j + O(h^{p_{N+1}}) \tag{3}$$

where $e_j \in D$ do not depend on h and $p_j > p_k > 0$ if $j > k$. For given $h_1, r_1, r_2, \ldots, r_{N-1}$ $(0 < r_j < 1)$ define $h_j = h_{j-1} r_{j-1}$. Assume that, if $j \geqq k$ then $\Delta_{h_j} v_1 = \Delta_{h_j} v_2$ implies $\Delta_{h_k} v_1 = \Delta_{h_k} v_2$. Let $\{\psi_j\}$ be a family of linear operators (depending on h_1) defined by:
(i) $\forall V \in D_{h_j}$, $\psi_j(V) = \Delta_{h_1}(v)$, where $v \in \Delta_{h_j}^{-1}(V)$. Assume also that $\|\psi_j\| \leqq 1$. Then we can define the following algorithm:

$$U_i^{(0)} = \psi_i\, U(h_i),\quad T_i^{(0)} = \Delta_{h_1} u^* - U_i^{(0)},\quad g_{\nu,i}^{(0)} = \rho_{\nu,i}^{(0)} = \tau_{\nu,i}^{(0)} = 1$$

$$(\nu, i = 1, \ldots, N)$$

$$U_i^{(k)} = \frac{\rho_{k,i-1}^{(k-1)}\, r_{i-1}^{p_k}\, U_{i-1}^{(k-1)} - U_i^{(k-1)}}{\rho_{k,i-1}^{(k-1)}\, r_{i-1}^{p_k} - 1},\quad g_{\nu,i}^{(k)} = \frac{\tau_{\nu,i-1}^{(k-1)}\, r_{i-1}^{p_k - p_\nu} - 1}{\rho_{k,i-1}^{(k-1)}\, r_{i-1}^{p_k} - 1} \cdot g_{\nu,i}^{(k-1)}$$

$$\rho_{\nu,i}^{(k)} = \frac{g_{\nu,i+1}^{(k)}}{g_{\nu,i}^{(k)}},\quad \tau_{\nu,i}^{(k)} = \frac{\rho_{k+1,i}^{(k)}}{\rho_{\nu,i}^{(k)}} \qquad (i, \nu = k+1, \ldots, N)$$

<u>Theorem</u>: If the ratios r_j $(j = 1, \ldots, N-1)$ are fixed then

$$T_i^{(k)} = \Delta_{h_1} u^* - U_i^{(k)} = \Delta_{h_1} \sum_{\nu=k+1}^{N} e_\nu\, g_{\nu,i}^{(k)}\, h_i^{p_\nu} + 0(h_1^{p_N+1})$$

$$(k = 0, \ldots, N-1;\ i = k+1, \ldots, N)$$

where the $g_{\nu,i}^{(k)}$ do not depend on h_i.

The error of discretization after $N-1$ steps is

$$\Delta_{h_1} u^* - U_N^{(N-1)} = T_N^{(N-1)} = e_N\, g_{N,N}^{(N-1)}\, h_N^{p_N} + 0(h_1^{p_N+1})$$

with

$$g_{N,N}^{(N-1)} = \prod_{k=1}^{N-1} \frac{\tau_{N,N-1}^{(k-1)}\, r_{N-1}^{p_k - p_N} - 1}{\rho_{k,N-1}^{(k-1)}\, r_{N-1}^{p_k} - 1}$$